Universality in the equilibration of quantum systems after a small quench

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A sudden change in the Hamiltonian parameter drives a quantum system out of equilibrium. For a finite-size system, expectations of observables start fluctuating in time without converging to a precise limit. A new equilibrium state emerges only in the probabilistic sense, when the probability distribution for the observable expectations over long times concentrates around their mean value. In this paper we study the full statistic of generic observables after a small quench. When the quench is performed around a regular (i.e., noncritical) point of the phase diagram, generic observables are expected to be characterized by Gaussian distribution functions ("good equilibration"). Instead, when quenching around a critical point a new, universal, double-peaked distribution function emerges for relevant perturbations. Our analytic predictions are numerically checked for a nonintegrable extension of the quantum Ising model.

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I. INTRODUCTION

Imagine preparing a closed quantum system in a given initial state ρ_0 and letting it evolve freely. After a sufficiently long time, an equilibrium, average state $\overline{\rho}$ emerges. Because of the unitary nature of the dynamics, in a finite system, the evolved state $\rho(t)$ cannot converge to $\overline{\rho}$ either in the strong or in the weak topology.[1] Equilibration in isolated quantum systems only emerges in a probabilistic fashion. We say that the observable O equilibrates to $\overline{O}$ if the expectation value $\langle O(t)\rangle$ is close to its average $\overline{O}$ most of the time. In other words, $\langle O(t)\rangle$ is seen as a random variable equipped with the (uniform) measure dt/T in the interval $t \in [0, T]$, where T is the total observation time which will be sent to infinity. The probability distribution of O is $P(o) := \overline{\delta(o - \langle O(t)\rangle)}$, where the bar refers to temporal averages: $\overline{f} := \lim_{T\to\infty} T^{-1}\int_0^T f(t)dt$. Broadly speaking, concentration phenomena for $P(o)$ correspond to quantum equilibration. The average value of a generic observable is readily obtained as $\overline{O} := \overline{\langle O(t)\rangle} = \mathrm{tr}(\overline{\rho} O)$, an equation that defines the equilibrium state to be $\overline{\rho} = \overline{\rho(t)}$. Equilibration, however, is related to the concentration of the distribution $P(o)$, a convenient definition of which is encoded in the variance ΔO^2. In Refs. [1] and [2] it was shown that the variance of any observable is bounded by the purity of the equilibrium state $\mathcal{P}(\overline{\rho}) := \mathrm{tr}(\overline{\rho}^2)$: This is an encouraging result: if $\mathcal{P}(\overline{\rho})$ is small, one has equilibration for every observable. Equilibration should depend on the dynamic and, possibly, on the initial state, not on the specific observable.

A convenient setting for probe quantum equilibration is that of a sudden quench. The system is initialized in the ground state of some Hamiltonian H_1 and then evolved unitarily with a small perturbation, $H_2 = H_1 + \delta\lambda V$. This situation is compelling from both a theoretical and an experimental point of view, thanks to the recent advances in cold atom technology [3–5].

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[1]For a different point of view see [28].

In this paper we analyze the full statistic of a generic observable $P(o)$ after a small quench. For small quenches performed around a regular (i.e., noncritical) point, the expected distribution $P(o)$ is Gaussian in the generic case. Equilibration is achieved in a standard fashion. Instead, for quenches performed around a critical point the distribution of generic observables tends to a new, universal, double-peaked function which we are able to compute.

This behavior was first demonstrated in [6] for a particular observable [the Loschmidt echo (LE)] on the hand of an exactly solvable model (Ising model in a transverse field). Here we show that the scenario first advocated in [6] is in fact general to small quenches for sufficiently relevant perturbations.

II. CRITICAL SCALING OF THE TIME-AVERAGED STATE

Here we consider the equilibrium distribution for small quench. When the quench is small one can either expand the eigenvectors of the evolution Hamiltonian H_2 with perturbation $+\delta\lambda V$ or expand the initial state with respect to a perturbation $-\delta\lambda V$. We take the latter point of view. Let the $t > 0$ Hamiltonian be $H_2 = \sum_n E_n |n\rangle\langle n|$. The initial state $|\psi_0\rangle$ is the ground state of $H_1 = H_2 - \delta\lambda V$. Then

$$|\psi_0\rangle = |0\rangle + \delta\lambda \sum_{n\neq 0} \frac{\langle n|V|0\rangle}{E_n - E_0}|n\rangle + O(\delta\lambda^2)$$

(note the plus sign in V). If the spectrum is nondegenerate, the equilibrium state has the form $\overline{\rho} = \sum_n p_n |n\rangle\langle n|$ [1,2,6]. The weights, up to second order in the quench potential, are given by

$$\begin{aligned} p_0 &= |\langle 0|\psi_0\rangle|^2 = 1 - \delta\lambda^2 \sum_{m\neq 0} \frac{|\langle m|V|0\rangle|^2}{(E_m - E_0)^2}, \\ p_n &= |\langle n|\psi_0\rangle|^2 = \delta\lambda^2 \frac{|\langle 0|V|n\rangle|^2}{(E_0 - E_n)^2}, \quad n \neq 0. \end{aligned} \tag{1}$$

Note that, up to the same order, the purity of the equilibrium state is given by $\mathrm{tr}(\overline{\rho}^2) = p_0^2$. The weight p_0 is precisely the square of the well-studied ground-state fidelity $F = |\langle 0|\psi_0\rangle|$

[7–11], and its scaling properties are well known [12]. If the perturbing potential is extensive and the quench is done around a regular (i.e., noncritical) point, $F \sim \exp(-\text{const} \times \delta\lambda^2 L^d)$, where d is the spatial system dimension. Instead, for quenches at a critical point $F \sim \exp[-\text{const} \times \delta\lambda^2 L^{2(d+\zeta-\Delta_V)}]$, ζ is the dynamical critical exponent and Δ_V is the scaling dimension of the perturbation V. Indeed it is intuitively clear that by shrinking $\delta\lambda$ at will, one should be able to transfer most of the spectral weight to p_0, a limit in which the purity is large. The preceding scalings tell us that we must have $\delta\lambda \ll L^{-Q}$ with $Q = d/2$ ($Q = d + \zeta - \Delta_V = 1/\nu$) in the regular (critical) case. These are the regimes of small quench characterized by a large purity and, hence, large variances for generic observables: in other words, poor equilibration.

However, the distributions of the p_n for critical and regular quenches are radically different. As we will see, this has direct consequences for the *general form* of the distribution of generic observables.

In the case of a critical quench there exist modes with vanishing energy: $E_k - E_0 = v k^\zeta$, where k now is a quasimomentum label. According to Eq. (2) the corresponding weight p_k becomes large and might even (apparently) diverge when $k \to 0$. In a finite system with periodic boundary conditions the momenta are quantized as $k = 2\pi n/L$, then one would infer that, for a certain weight, $p_1 \sim \delta\lambda^2 L^{2\zeta}$. This, however, is not the correct scaling, as we did not include the scaling of the matrix element. To find the exact scaling we can reason as follows. Define the functions $M(E_n) := |\langle 0|V|n\rangle|^2$, and $p(E_n) := p_n$. With the help of the density of states $\rho(E) = \text{tr}\delta(E - H)$, one can write the fidelity susceptibility χ as

$$\chi = \sum_{m\neq 0} \frac{|\langle m|V|0\rangle|^2}{(E_m - E_0)^2} = \int_{E_1}^{E_{\max}} \frac{M(E)}{(E - E_0)^2}\rho(E)dE. \quad (2)$$

We are interested in the scaling properties of $M(E)$ after a rescaling of the energy. At criticality it is natural to assume that $M(E)$ is a homogeneous function at the lower edge: $M(E) \sim (E - E_0)^\alpha$. Instead, the product $\rho(E)dE$ is invariant under rescaling of the energy. The scaling of the fidelity susceptibility is known [12]: $\chi \sim L^{2(d+\zeta-\Delta_V)} \sim E^{-2(d+\zeta-\Delta_V)/\zeta}$, so from $\chi \sim E^{\alpha-2}$, we obtain $\alpha = 2(\Delta_V - d)/\zeta$. Using the fact that, for the operator driving the transition $\Delta_V = d + \zeta - 1/\nu$ [13], we obtain

$$p(E) \sim \delta\lambda^2 E^{-2/(\zeta\nu)}. \quad (3)$$

In this equation the energy is measured from the ground state, so that, being the system critical, E can be arbitrarily close to zero in the large size limit. The prediction, Eq. (3), agrees with an explicit calculation of the quantum Ising model ($p(\omega) = 2c(\omega)$ in [6]).

As a by-product of this analysis we obtain $\langle 0|V|k\rangle \sim L^{d-\Delta_V} = L^{-\zeta+(1/\nu)}$. Note that here V is the extensive perturbation. If $V = \sum_x V(x)$, for the intensive component we get

$$\langle 0|V(x)|k\rangle \sim L^{-\Delta_V} = L^{-\zeta-d+(1/\nu)}. \quad (4)$$

Equation (4) is in agreement with the analysis in Refs. [14–16] performed on the sine-Gordon model. In that case $d = \zeta = 1$ and one gets $\langle 0|\cos[\beta\phi(x)]|k\rangle \sim L^{-2+(1/\nu)}$. In fact formula (12) in Ref. [16] can be written as $\langle 0|\cos(\beta\phi(x))|k\rangle \sim L^{-K}$, where $K = 2 - (1/\nu)$ is the scaling dimension of the cosine term.

The content of Eq. (3) is the following. For a relevant perturbation ($d + \zeta > \Delta_V$) of a critical point some spectral weights p_n tend to be large. At finite size, the lowest modes have energy $E_n = v(2\pi n/L)^\zeta$, so that $p_n \sim \delta\lambda^2 L^{2/\nu}$. In practice, since in the region of validity of perturbation theory, p_0 is already "large," the sum rule $\sum_n p_n = 1$ constrains to have only very few p_n appreciably different from zero. We expect this scenario to be more pronounced for strongly relevant perturbations, in other words, when the exponent $2/\nu$ is large. When this is the case, the sum rule can be saturated by taking a very small number of terms $n_{\max}$: $1 = \sum_n p_n \approx \sum_{n=0}^{n_{\max}-1} p_n$. In our numerical simulations (see the following) we have verified that, for a case with $\nu = 1$, the sum rule is already saturated by taking as little as three terms, that is, $n_{\max} = 3$. Moreover, most of the weight is split between p_0 and p_1, while p_2 is already orders of magnitude smaller.

The same considerations can clearly be drawn for the amplitudes $c_n = \langle n|\psi_0\rangle = \delta\lambda\langle n|V|0\rangle/(E_0 - E_n) + O(\delta\lambda^2)$ for $n > 0$, for which $p_n = |c_n|^2$. Defining the function $c(E_n) = c_n$ with the same reasoning as used previously, one sees that, for $E \to 0$, $c(E) \sim \delta\lambda E^{-1/(\zeta\nu)}$. Alternatively, for some low-lying excitations with quasimomentum k, $c_k = \langle k|\psi_0\rangle \sim \delta\lambda L^{1/\nu}$. Since $c(E)$ is a rapidly decreasing function, and because of the sum rule for the c_n, one obtains a good approximation for the time-evolved wave function by just resorting to very few, $n_{\max}$, amplitudes: $|\psi(t)\rangle \approx \sum_{n=0}^{n_{\max}-1} c_n e^{-itE_n}|n\rangle$.

III. EQUILIBRIUM DISTRIBUTION FOR SMALL QUENCHES

Let us now illustrate the consequences of these findings on the equilibration. Consider the time evolution of a generic observable $\langle O(t)\rangle$. We also give results for the LE, as it is attracting an increasing amount of attention [17–23]. The Loschmidt echo is defined as $\mathcal{L}(t) = |\langle\psi_0|e^{-itH_2}|\psi_0\rangle|^2$. Note that, as pointed out in [6], the LE can be written as the expectation value of a particular observable $\langle O_{\mathcal{L}}(t)\rangle$, with $O_{\mathcal{L}}$ given by $O_{\mathcal{L}} = |\psi_0\rangle\langle\psi_0|$. Expanding $\mathcal{L}(t)$ and $\langle O(t)\rangle$ in the eigenbasis of H_2, we obtain

$$\mathcal{L}(t) = \overline{\mathcal{L}} + \sum_{m>n} 2p_n p_m \cos[t(E_n - E_m)], \quad (5)$$

$$O(t) = \overline{O} + \sum_{n\neq m} \langle n|O|m\rangle c_m \overline{c_n} e^{-it(E_m - E_n)}$$
$$= \overline{O} + \sum_{n>m} 2\langle n|O|m\rangle c_m c_n \cos[t(E_m - E_n)], \quad (6)$$

where in the last line we assumed that both the observables and the wave functions are real, as happens in most cases. As we have seen, for a small quench around criticality both c_n and p_n will be rapidly decreasing after their maximal value (in modulus), and a good approximation to Eqs. (5) and (6) can be obtained by retaining only a few terms. We have observed that the following minimal prescription retaining only the three

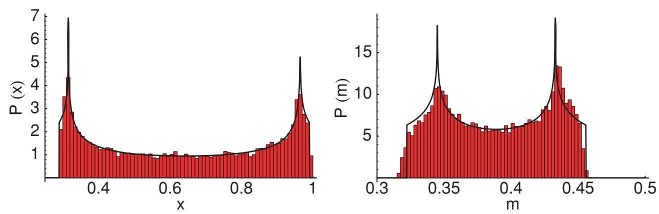

FIG. 1. (Color online) Probability distributions for a small quench around a critical point. $P(x) = \overline{\delta(x - \mathcal{L}(t))}$ and $P(m) = \overline{\delta(m - \langle\sigma_1^z(t)\rangle)}$ refer to the Loschmidt echo (left) and magnetization (right), respectively. The thick lines were obtained using the prediction of Eq. (7), with only the three largest weights. Note the large spread of the distributions compared with their total support: $P(x) \in [0, 1]$ and $P(m) \in [-1, 1]$. Parameters $L = 16$, $\kappa_1 = \kappa_2 = 0.4$, $h_1 = 0.218$, and $\delta h = h_2 - h_1 = 0.04$ are close to criticality (see [24]). Data were obtained by Lanczos diagonalization of Eq. (8), keeping as many lowest-energy vectors as necessary for the sum rule $\sum_{n=0}^{n_{\max}-1} p_n \simeq 1$ to be satisfied within the prescribed accuracy.

largest components works fairly well:

$$F(t) = \overline{F} + A\cos(\omega_A t) + B\cos(\omega_B t). \tag{7}$$

For instance, $A = 2p_0 p_1$ and $B = 2p_0 p_2$ for the LE, while $A = 2O_{0,1}c_0 c$ and $B = 2O_{0,2}c_0 c_2$ for a more generic observable O. The distribution function related to the time signal, Eq. (7), $P(f) = \overline{\delta(f - F(t))}$, has been computed exactly in Ref. [6]. $P(f)$ is a symmetric function around the mean $\overline{F}$ supported in $[\overline{F} - ||A| + |B||, \overline{F} + ||A| + |B||]$, with logarithmic divergences at $f = \overline{F} \pm ||A| - |B||$ (see Fig. 1).

This scenario can be summarized as follows: For a small quench around a critical point, generic observables equilibrate only very poorly. The distribution function for a generic observable is a double-peaked distribution with a relatively large mean, a behavior completely different from the Gaussian one.

To complete the analysis let us now discuss the case of a small quench in a regular point of the phase diagram. At regular points there are no gapless excitations and the weights are bounded by $p(E) \leqslant M(E)/\Delta^2$, where Δ is the smallest gap. Since the theory is not scale invariant, $M(E)$ will not be a homogeneous function and, in particular, will not display any singularity. The picture, then, is the following: In the perturbative regime ($\delta\lambda^2 L^d \lesssim 1$) we still have a "large" lowest weight, but besides p_0, no other p_n dominates, and the sum rule $\sum_n p_n = 1$ is saturated only recurring to a relatively large bunch of p_n values.

In general, predicting the precise behavior of observables in this case will be difficult, as one needs to have knowledge of many different weights in Eqs. (5) and (6). However, we can give a simple argument to expect a Gaussian behavior for the *generic* case. As we have argued, the sum in Eq. (6) now contains many terms. If the energy differences $E_n - E_m$ are rationally independent, along the time evolution, each variable $X_{n,m} := 2\langle n|O|m\rangle c_m c_n$ will span uniformly the interval $[-2|\langle n|O|m\rangle c_m c_n|, 2|\langle n|O|m\rangle c_m c_n|]$. As long as the variables $X_{n,m}$ can be considered *independent*, $O(t) - \overline{O}$ can be thought of as a sum of independent random variables. Since, as we have seen, the sum is made over many variables, the central limit theorem applies and the resulting distribution will be Gaussian. This argument can fail when the variables cannot be taken as independent. This can happen, for instance, when a certain observable is pushed toward its maximum or minimal value by the action of some field. Consider, for example, the case of a transverse magnetization σ_j^z in the presence of a high field $-h\sum_i \sigma_i^z$. For increasing h the mean of $\langle\sigma_j^z(t)\rangle$ will be pushed toward 1. Since $\langle\sigma_j^z(t)\rangle$ is supported in $[-1, 1]$, the corresponding distribution can cease to be Gaussian as its mean is pushed against the (upper) border of its support. In this case the distribution function will look like a "squeezed" Gaussian. A similar effect was observed in Ref. [6] to take place in the LE when the system size became the largest scale of the system. In any case, however, if the variables cannot be considered as independent, any possible distribution function (and not only a squeezed Gaussian) can arise.

IV. NUMERICAL TEST

We now check our predictions on the hand of a nonintegrable model. As a test model we chose to use the so-called TAM Hamiltonian (transverse axial next-nearest-neighbor Ising model). The Hamiltonian is

$$H = -\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x - \kappa\sigma_i^x\sigma_{i+2}^x + h\sigma_i^z\right), \tag{8}$$

and periodic boundary conditions are used ($\sigma_{L+i}^x = \sigma_i^x$). A positive κ frustrates the order in the σ^x direction. The reason for our choice is, at least, twofold. (i) The TAM is a nonintegrable generalization of the one-dimensional quantum Ising model for which results are already available [6]. (ii) The model, Eq. (8), has only a discrete $\mathbb{Z}_2$ symmetry ($P_z = \prod_i \sigma_i^z$); consequently, the ground state lives in a large $d_{\mathrm{GS}} = 2^{L-1}$ dimensional space. In practice, d_{GS} is the effective Hilbert space dimension, and we would like it to be as large as possible. For instance, after a quench the purity of the equilibrium state is bounded by $\mathrm{tr}(\overline{\rho}^2) \geqslant d_{\mathrm{GS}}^{-1}$. This is to be contrasted with other models used in the literature with larger symmetry groups [i.e., SU(2)], for which the dimension of the block containing the ground state is still exponential in L but considerably reduced with respect to that of the full Hilbert space 2^L.

The model, Eq. (8), displays four phases (see, e.g., Refs. [24–27], and references therein): ferromagnetic, $++++$; antiphase, $++--$; paramagnetic; and a floating phase with algebraically decaying spin correlations. In particular, for small frustration $\kappa \leqslant \frac{1}{2}$, upon increasing the external field h there is a transition from ferromagnetic to paramagnetic. This transition is believed to fall in the Ising universality class, and so the critical theory is described by a conformal field theory with central charge $c = \frac{1}{2}$ and $d = \zeta = \nu = 1$. We performed our numerical simulation for the critical quench on this critical line.

We illustrate our findings for two particular yet physically well motivated observables: the LE, $\mathcal{L}(t) = |\langle\psi_0|e^{-itH_2}|\psi_0\rangle|^2$, and the transverse magnetization $m(t) = \langle\psi_0(t)|\sigma_i^z|\psi_0(t)\rangle$.

Since $d = \zeta = \nu = 1$, according to Eq. (3), we expect a strong divergence at low energy: $p(E) \sim E^{-2}$. Consequently we expect very few p_n, $n > 0$, to have non-negligible weight

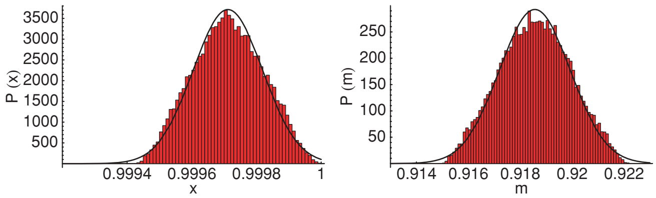

FIG. 2. (Color online) Probability distributions for a small quench around a regular point. $P(x) = \overline{\delta(x - \mathcal{L}(t))}$ and $P(m) = \overline{\delta(m - \langle\sigma_1^z(t)\rangle)}$ refer to the Loschmidt echo (left) and magnetization (right), respectively. The thick lines are Gaussian, with the same mean and standard deviation. The quench was performed in the paramagnetic phase; the parameters are $L = 12, \kappa_1 = \kappa_2 = 0.3, h_1 = 1.4$, and $\delta h = 0.04$. Note the very small spread of the distributions. Data were obtained by full diagonalization of the Hamiltonian, Eq. (8). Histograms were obtained by sampling 40 000 random times in an interval $t \in [0, T]$ with $T = 16\,000$.

and, so, Eq. (7) to be a valid approximation. Indeed the results based on numerical diagonalization compare well with the prediction based on Eq. (7) (Fig. 1). Note the very large spread of the distributions compared to their total support: "poor equilibration."

For comparison we performed a similar numerical simulation for a small quench in a regular point of the phase diagram. As expected the resulting distribution functions are approximately Gaussian (Fig. 2). Note the very small variances of the distributions already for a relatively short size: "good equilibration."

V. CONCLUSIONS

In this paper we have investigated the detailed structure of equilibration after a small quench; that is, the system is initialized in the ground state of a given Hamiltonian H_1 and then let to evolve with a slightly perturbed Hamiltonian, $H_2 = H_1 + \delta\lambda V$. In the limit $\delta\lambda \to 0$ equilibration is trivial in that, for all observables, $P(o) = \overline{\delta(o - \langle O(t)\rangle)}^t = \delta(o - \langle O\rangle)$. However, this limit is approached very differently depending on whether or not the Hamiltonian H_1 is critical. For quenches around a regular point of the phase diagram the expected distribution for generic observables is a Gaussian one. Equilibration arises in the most standard fashion. Instead, for small quenches around a critical point the situation is radically different. The distribution function for generic observables $P(o)$ tends to a universal double-peaked function for relevant perturbations.

The key step in obtaining these results is to characterize the overlaps $c_n = \langle n|\psi_0\rangle$ between the initial state $|\psi_0\rangle$ and the quenched Hamiltonian eigenstates $|n\rangle$. We have shown that, at criticality, the function $c(E_n) = c_n$ (where E_n is the eigenenergy) decays very rapidly, $c(E) \sim E^{-1/(\zeta\nu)}$, and this in turn generically implies the observed double-peaked distributions. The analytical predictions have been checked numerically on the hand of a nonintegrable extension of the quantum Ising model.

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