

Undecidability in tensor network states

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Recent work has examined how undecidable problems can arise in quantum-information science. We augment this by introducing three undecidable problems stated in terms of tensor networks. These relate to ideas of Penrose about the physicality of a spin network representing a physical process, closed timelike curves, and Boolean relation theory. Seemingly slight modifications of the constraints on the topology or the tensor families generating the networks lead to problems that transition from decidable to undecidable to even always satisfiable.

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As the limitations of computers are ultimately governed by the laws of physics, and as physical process can in turn be viewed as computations, it is becoming increasingly important to understand how to bridge computer science and physics. In this setting, quantum mechanics can be thought of as a generalization of classical computation. Most work has focused on developing quantum-complexity theory and algorithms governed by quantum theory. Building on these successes, increasingly subtle ideas in computer science are finding their way into quantum physics. An emerging theme in this regard is decidability in quantum information [1–3], wherein some undecidable problems in quantum-information theory were discovered.

A tensor network state is a quantum state represented by a tensor contraction network [4]; see [5] for a review. These networks constitute a graphical representation which can be thought of as quantum circuits augmented with preparations and postselection [6]. We augment recent decidability work [1,3] with an emphasis on finding additional problems naturally phrased in terms of tensor network states. We address three questions:

Question 1. Is every tensor network built from a library \mathcal{F} of n at most d -qubit quantum gates, chained together and allowing one postselection per gate, physical?

For large enough n and d , this question is undecidable. Theorems 1 and 2 offer alternative formulations of this result, while Theorem 4 directly addresses Question 1. Suppose one writes down a tensor network using some fixed library of primitive operations, each constructed from some combination of unitary gates, state preparations, and postselected measurements. It may be that the resulting network is physically impossible. That is, although each individual operation in the network would be allowed in certain circumstances, their combination would be impossible.

One way to prove this is by considering bell states and postselected bell costates. We can then form postselected closed timelike curves (P-CTC) [7]. Thus we can obtain a zero tensor by inadvertently creating a grandfather paradox [see the elementary example of a grandfather-type paradox in

Fig. 1(a)]. We show that determining whether or not we can do so is undecidable.

Theorem 1. Determining whether we can construct a tensor network which contains a hidden-grandfather-paradox type contradiction is undecidable.

Note that the question addressed is not whether exactly this loop exists, but whether there is *some* contradiction which, like the grandfather paradox, results in a tensor network with value zero, and hence a situation which the theory predicts is physically impossible. In general such paradoxes or contradictions will be an intricate chain of events too complex to follow by hand. The grandfather paradox provides a clear example of such a contradiction where the contradiction is easy to see.

The *rank* of a tensor is the least r such that it can be expressed as a sum of r rank-one (product) tensors. Hastad showed that given a tensor T described by a table of numbers, the associated tensor rank decision problem “rank(T) $\leq r$?” is NP-hard [8]; in fact most tensor problems are NP-hard [9]. Theorem 4 implies that given a library \mathcal{F} of tensors, the question of whether we can construct a tensor network state Ψ on n qubits such that rank(Ψ) $\leq r$ is undecidable. Hence,

Theorem 2. The constructibility of rank-bounded tensors is undecidable.

To examine carefully the boundary between decidability and undecidability, we consider the subclass of Boolean tensor network states. These give insight into the general situation as the image of the support map. We address two questions:

Question 2. Given a finite library \mathcal{F} of tensors, can a given Boolean tensor network state be built from \mathcal{F} ?

Question 3. Given a finite library \mathcal{F} of tensors including swap and fanout, can a given Boolean tensor network state be constructed from \mathcal{F} ?

When swap and fanout are not included (as in Question 2), the problem is undecidable. In contrast to this, and to illustrate the subtlety of the issue, Corollary 2 explains how including two specific tensors in \mathcal{F} means that a quadratic algorithm exists to solve Question 3.

This work is organized as follows. In Sec. I we restate a not-well-known early conjecture by Penrose relating to the physicality of a tensor network representation of a quantum process. This leads to a discussion of tensor networks with

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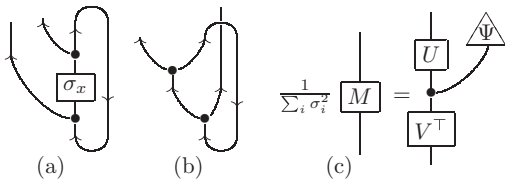


FIG. 1. (a), (b) Closed timelike curves. Grandfather paradox (a) and unproved theorem paradox (b) tensor networks from [7], with value 0 (left) and $(1/\sqrt{2})(|00\rangle + |11\rangle)$ (right). The problem of determining whether a tensor network can be constructed from a set of generators such that it contains a grandfather-paradox-type contradiction (is the zero tensor) is undecidable. The filled dot denotes a copy tensor; in the qubit case, the copy tensor is $\text{CNOT} \circ (|0\rangle \otimes \text{id})$. (c) Quantum singular-value decomposition to implement an $n \times n$ integer matrix up to global rescaling. The wire ending in the measurement Ψ has n states and the measurement is $|\Psi\rangle = (\sum_i \sigma_i^2)^{-1} \sum_i \sigma_i |i\rangle$ where σ_i are the singular values of M .

postselected outputs in Sec. II. Interestingly, many of the results hold when restricted to the class of Boolean quantum states. In Sec. III we study this class, first showing that many problems in this class are naturally decidable, including connecting the satisfiability problem with the physicality of a quantum state. Seemingly slight modifications result in a host of undecidable problems related to Boolean tensor network states.

I. TENSOR NETWORKS AND THE GRANDFATHER PARADOX

A grandfather-type paradox can result from loops in a tensor network, enabling contradictions. The tensor network formalism predicts the same result for the grandfather paradox and unproved theorem paradox tensor networks as was obtained mathematically and experimentally in [7], rather than the result that would accord with Deutsch's interpretation of the grandfather paradox network in terms of a mixed-state fixed point $\rho = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ [10]. Thus [7] provides experimental evidence for the tensor network formalism built from unitary operators, preparations (e.g., into maximally entangled states), and postselection—in particular, that a tensor network is a zero tensor if and only if it is forbidden by physical law. The following was argued by Penrose in 1967.

Theorem 3 (Penrose, 1967 [11]). The norm of a spin network vanishes if and only if the physical situation it represents is forbidden by the rules of quantum mechanics.

Example 1 (examples of Penrose's theorem). Consider a Bell state $\Phi^+ = |00\rangle + |11\rangle$. The amplitude of the first party measuring $|0\rangle$ followed by the second party measuring $|1\rangle$ is zero. This vanishing tensor network contraction is given by $\langle 01, \Phi^+ \rangle$. For a second example, consider the norm of a network of connected tensors thought to represent a valid state $|\psi\rangle$. If the norm found from contracting the state with a conjugated copy of itself $\langle \psi, \psi \rangle$ vanishes, the network necessarily represents a nonphysical quantum state, by Penrose's theorem. We give examples below.

II. PHYSICALITY OF A TENSOR NETWORK WITH POSTSELECTION

First we consider questions of measurement occurrence (as in [1]). Suppose a candidate state is described as a tensor network, in terms of locally physical operations. It is natural to consider whether such a candidate state is physical. A library of tensors can be composed by contraction to form a class. To prove that the physicality question is undecidable for a given class of tensor networks, we encode a known undecidable problem ([12] surveys these). The undecidable question then maps to the constructibility of a network with vanishing norm (evaluated by contraction with a copy of itself).

Of course, the word problem for a finite subgroup of the group of unitary $n \times n$ matrices is decidable by multiplying out the words. The matrix mortality problem is also straightforward. Consider the question of whether there exists a nonphysical (zero-norm) circuit built by chaining single-qutrit gates together with an initial state and a measurement. This is shown to be undecidable in [2] by directly embedding Post's correspondence problem.

Remark 1 (the word problem for linear groups is decidable). Consider a group given as a (possibly infinite) list of generators and relations. *The word problem for groups* asks whether two words in the generators represent the same group element. Even if we restrict to finitely presented (but not finite) groups, this problem is undecidable. Thus one might hope that a faithful unitary representation would lead to an undecidable word problem for quantum circuits. However, not all finitely presented groups (or even residually finite groups) have a faithful representation in $\text{GL}(n, \mathbb{K})$ for some n and field \mathbb{K} . Those that do are called *linear*, and have solvable word problems: Given words σ, τ one can simply multiply out the matrices and check whether $\sigma \tau^{-1}$ is the identity. Alternatively, given a tensor with an input and output wire, one could bend one of the wires to form a bell state or costate. Checking whether the corresponding tensor network state is the generalized bell state is decidable. In fact, the word problem for linear groups in characteristic zero is solvable in log space [13]. See [14] for background on such complexity classes.

This yields the following observation:

Observation 1. The word problem for quantum circuits built only from unitary gates (no measurements or comeasurements) is solvable in log space and the mortality problem is trivial; i.e., all such circuits are physical.

In contrast, in Theorem 4 we will show that if measurement and postselection is included, the physicality problem becomes undecidable. First we give a simple construction that will be needed in the proof of undecidability. Let $\|M\|_F$ be the Frobenius norm.

Lemma 1. Let M be an $n \times n$ integer matrix. Then $\|M\|_F^{-2} M$ can be implemented using $m = \lceil \log_2 n \rceil$ -qubit unitary gates, m CNOT (=controlled-NOT) gates, $\binom{m}{2}$ SWAP gates, and one m -qubit postselection operation.

The CNOT and SWAP gates are used to create a generalized copy tensor.

Proof. Given an integer matrix M , write its singular-value decomposition $M = U \Sigma V^T$; we can take U, V unitary (in

fact real orthonormal) and implement Σ by a copy tensor and postselection Ψ as in Fig. 1(c). ■

For example, a two-qubit copy tensor used as a fanout can be implemented as $(\text{id} \otimes \text{SWAP} \otimes \text{id}) \circ \{[\text{CNOT} \circ (\langle 0| \otimes \text{id})]^{ \otimes 2}\}$. Given a 3×3 integer matrix M , encode it as the two-qubit (nonunitary) operator M' as follows:

$$M = \begin{pmatrix} p & 0 & 0 \\ 0 & r & 0 \\ q & s & 1 \end{pmatrix} \quad M' = \begin{matrix} & \langle 00| & \langle 01| & \langle 10| & \langle 11| \\ \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} & \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ q & s & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

It is sufficient for undecidability [15] to consider matrices of the form M , $p, r > 0$, $q, s \geq 0$ together with the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A' = \begin{matrix} & \langle 00| & \langle 01| & \langle 10| & \langle 11| \\ \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

We have the singular-value decomposition $A' = U \Sigma V^T$. So the postselection implementing Σ will be given by $|\Psi_A\rangle = \frac{1}{\sqrt{5}}(2|00\rangle + |01\rangle)$.

Now we can establish the undecidability of Question 1 for large enough n and d by reducing to the matrix mortality problem.

Theorem 4. When $n \geq 8$ and $d \geq 2$, or $n = 2$, $d \geq 5$, Question 1 is undecidable.

Proof. Question 1 asks whether, given a set of (postselection-including) operators M_1, M_2, \dots, M_n , there is a finite word such that $M_{i_1} M_{i_2} \dots M_{i_k} = 0$. The vanishing of the morphism defined by the contraction corresponding to the word means the sequence is nonphysical.

By Lemma 1, we may assume our operators M_i are integer matrices. Then applying results on the integer matrix mortality problem gives undecidability for our problem. In [15], it was established that eight 3×3 integer matrices sufficed for undecidability, and we show above how to embed this into two-qubit operators. In [16] it was shown that two 24×24 integer matrices also suffice, and these can be embedded in five-qubit operators, which gives the second part of the result. ■

We have so far been considering decidability in tensors without restriction on the components. It turns out that many decidability results can be obtained when considering the subclass of Boolean tensor networks. These also afford an illuminating connection to solving satisfiability instances, by tensor network contraction.

III. BOOLEAN TENSOR NETWORK STATES

A quantum state is *Boolean* if and only if it can be written in a local basis with amplitude coefficients taking only values 0 or 1. We relate such states to Boolean functions, allowing for tools from algebra to be applied to their analysis.

Remark 2 (notation). Suppose $x \in \{0, 1\}^n$ denotes an n -long Boolean bit string, and use $|x\rangle$ as an index for a basis state. If $f : \{0, 1\}^n \rightarrow \{0, 1\}$ then $|f(x)\rangle$ also indexes a basis state.

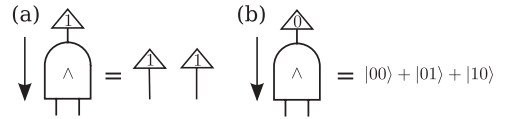
Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be any Boolean function. Then $|\psi_{\mathbb{B}}\rangle = \sum_{\mathbf{x}} |\mathbf{x}\rangle |f(\mathbf{x})\rangle$ is a representative in the class of Boolean states. Hence every Boolean function gives rise to a quantum

state. A *Boolean relation* R of arity n is a subset of $\{0, 1\}^n$ (for example, an OR clause in 3-SAT can be thought of as a relation $\text{OR} \subset \{0, 1\}^3$ as $\text{OR} = \{0, 1\}^3 \setminus \{111\}$). Every quantum state written in a local basis with amplitude coefficients taking values in $\{0, 1\}$ gives rise to a Boolean relation.

Theorem 5 (Boolean tensor network states). A tensor network representing a Boolean quantum state is determined from the classical network description of the corresponding relation.

Theorem 5 was given in [17], where the quantum tensor networks are found by letting each classical gate act on a linear space and changing the composition of functions to contraction of tensors. Raising or lowering an index transforms kets to bras or vice versa, given a basis.

Example 2 (AND-tensors). As an example of a Boolean tensor, consider the AND-tensor defined as $\text{AND}_{jk}^i = |00\rangle\langle 0| + |01\rangle\langle 0| + |10\rangle\langle 0| + |11\rangle\langle 1|$. We depict the contraction of the output of the AND-tensor with $|1\rangle$ and $|0\rangle$ as



In (a) the contraction results in the product state $|11\rangle$ and in (b) the contraction yields $|00\rangle + |01\rangle + |10\rangle$.

A. Satisfiability in Boolean tensor network states

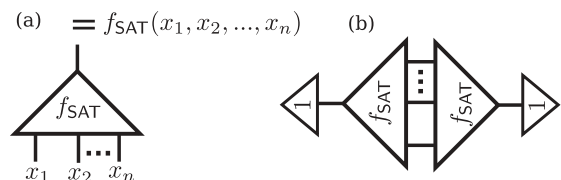
Theorem 6 (counting 3-SAT solutions). Let f be given to represent a 3-SAT instance. Then the standard two-norm length squared can be made to give the number of satisfying assignments of the instance.

Proof. The quantum state takes the form $|\psi_f\rangle = \sum_{\mathbf{x}} |\mathbf{x}\rangle |f(\mathbf{x})\rangle = \sum_{\mathbf{x}} f(\mathbf{x}) |\mathbf{x}\rangle$. We calculate the inner product of this state with itself $\|\psi_f\|^2 = \sum_{\mathbf{x}\mathbf{y}} f(\mathbf{x})f(\mathbf{y}) \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{\mathbf{x}} f(\mathbf{x})$ which gives exactly the number of satisfying inputs. This follows since $f(\mathbf{x})f(\mathbf{y}) = \delta_{\mathbf{x}\mathbf{y}}$. We note that for Boolean states, the square of the two norm equals the one norm. ■

Remark 3 (counting 3-SAT solutions). Solving the counting problem (6) is known to be #P-hard.

Corollary 1 (solving 3-SAT instances). The condition $\|\psi_f\| > 0$ implies that the 3-SAT instance corresponding to f has a satisfying assignment, which corresponds to an NP-complete decision problem.

Remark 4 (graphical depiction). Below (a) gives a network realization of the function and determining whether the network in (b) contracts to a value greater than zero solves a 3-SAT instance.



As we have considered simple restrictions on problems to transform a generally undecidable problem into a decidable one, we can also place suitable restrictions on satisfiability

problems such that they result in a class of only satisfiable instances.

Suppose a state is defined by a tree tensor network [5,18] such that the support of the self-contraction of tensor T on each node results in the identity map [i.e., $\text{Supp}(TT^\dagger) = \mathbf{1}$]. Then the CSP defined by that state is satisfiable.

B. Decidability in Boolean tensor network states

Now consider the *implementability* question: Given a Boolean tensor f , is there some wiring of a collection \mathcal{F} of Boolean tensors that produces it? This is a word problem for planar operads, and such problems are in general undecidable.

Theorem 7. Question 2 is in general undecidable.

Proof. We rely on the proof of undecidability of Question 2 for Boolean predicates by Cook and Bruck [19]. As long as we consider only Boolean predicates in our tensor network and do not introduce swap or fanout (copy tensor), the result applies directly here. ■

Represent a function $x: \{1, \dots, m\} \rightarrow R$ by a matrix with entries $x_j^i \in \{0, 1\}$, $i = 1, \dots, m$, $j = 1, \dots, n$. Given a Boolean function f of arity m , a Boolean relation R of arity n , and a function $x: \{1, \dots, m\} \rightarrow R$, denote by $f(x) \in \{0, 1\}^n$ the element $(f(x_1^1, \dots, x_1^m), f(x_2^1, \dots, x_2^m), \dots, f(x_n^1, \dots, x_n^m))$.

Definition 1. Let R be a Boolean relation of arity n and f a Boolean function of arity m . We say that R is an *invariant* of f if for any $x: [m] \rightarrow R$, $f(x) \in R$, and that f is a *polymorphism* of R if R is an invariant of f .

If S is a set of Boolean relations, let $\text{Pol}(S)$ be the set of polymorphisms of every relation $R \in S$. If B is a set of Boolean functions, let $\text{Inv}(B)$ be the set of invariants of every function $f \in B$. By [20,21], the co-clone of S is $\text{Inv}[\text{Pol}(S)]$ and the clone of B is $\text{Pol}[\text{Inv}(B)]$.

Theorem 8. Let \mathcal{F} be a set of Boolean tensors containing the copy tensor on three bits. Then the set of Boolean tensors constructible from \mathcal{F} is $\text{Inv}[\text{Pol}(\mathcal{F})]$.

Proof. By [20,21], the co-clone of \mathcal{F} is $\text{Inv}[\text{Pol}(S)]$. Thus it remains to show that the notion of a co-clone of a set of relations containing the all-equal relation, and constructible

tensor networks built from the corresponding Boolean tensors is the same. A co-clone is closed under the Cartesian product, identification of variables, and projection. In the category of relations, the Cartesian product is the monoidal (tensor) product. Closure under identification means connecting two wires, from the same or different tensors. Closure under projection is equivalent to connecting a plus state to a wire of the relation; this can be implemented by identifying two legs of a three-legged all-equal relation. ■

Corollary 2. There is a quadratic algorithm for Question 3.

Proof. The set of Boolean predicates implementable by wiring predicates in \mathcal{F} , assuming swap and fanout, is the co-clone generated by \mathcal{F} , $\text{Pol}[\text{Inv}(\mathcal{F})]$. There exists a *quadratic* algorithm determining whether a given Boolean relation r is in the co-clone generated by a finite set of relations [22]. ■

The qubit case is quite special. The equivalent implementability problem (for qudits) with $k > 2$ states is co-NEXPTIME-complete [23]. This further illustrates how transitions among satisfiable, polynomial-time, NP- and #P-complete, co-NEXPTIME-complete, and undecidable problems can be caused by seemingly small changes in the allowed constructions.

IV. CONCLUSION

The physical significance of decidability has been explored in recent work, and its relevance to quantum-information science is a topic of current discussion. We added to this discussion by focusing on decidability questions as they relate to tensor network states. We have tried to focus here on the subtle differences between an undecidable problem and the slight variation of this problem to transform it into a decidable one.

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