The algebra of Grassmann canonical anticommutation relations and its applications to fermionic systems

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We present an approach to a noncommutativelike phase space which allows to analyze quasifree states on the algebra of canonical anti-commutation relations (CAR) in analogy to quasifree states on the algebra of canonical commutation relations (CCR). The used mathematical tools are based on a new algebraic structure the "Grassmann algebra of canonical anticommutation relations" (GAR algebra) which is given by the twisted tensor product of a Grassmann and a CAR algebra. As a new application, the corresponding theory provides an elegant tool for calculating the fidelity of two quasifree fermionic states which is needed for the study of entanglement distillation within fermionic systems. © *2010 American Institute of Physics.* [doi:10.1063/1.3282845]

I. INTRODUCTION

Using anticommuting Grassmann variables for calculating physical quantities for fermionic systems is a well established technique. This concerns, in particular, the calculation of expectation values of quasifree (Gaussian) fermion states. The idea is to replace the linear combinations of canonically anticommuting Fermi field operators with complex coefficients by linear combinations with coefficients that are anticommuting Grassmann numbers. As a consequence, these linear combinations fulfill "canonical commutation relations." By interpreting tuples of Grassmann numbers as "phase space vectors," a similar analysis can be carried out as it is known for the bosonic case. In their article, Cahill and Glauber[4] used this technique to analyze density operators for fermionic states. These calculations are presented on a symbolic level by starting from a set of computational rules with less focus on the underlying mathematical structure.

On the other hand, there are several mathematically rigorous applications of the Grasmann calculus to fermion systems within the present literature. Examples are the treatment of perturbation theory for many fermion systems, as the Hubbard model for fermions,[15] as well as the analysis of fermion correlation functions within constructive quantum field theory and constructive renormalization (see Refs. 8, 17, and 19 and references given therein).

The aim of this paper is to provide the appropriate mathematical framework for a "quantum harmonic analysis on phase space" for fermion systems by using the Grasmann calculus. This can be seen as a noncommutative analog of the quantum harmonic analysis for bosons that is treated by Werner in Ref. 23. We show here a collection of basic propositions and theorems which help to simplify calculations and which also allows to confirm the results of Ref. 4 in a mathematically rigorous manner. In addition to that, our approach allows, up to certain extend, to consider also infinite dimensional systems.

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The idea to consider fermionic phase space methods has been developed in the context of supersymmetric quantum theories and has been studied by many authors (see, e.g., Refs. 5 and 9–12 and references given therein). Roughly speaking, the discussion of fermionic phase space in the present literature is used to build a "classical analog" of bosonic phase space and to perform its (second) quantization. Furthermore, most of the tools developed so far are used to study the aspects of supersymmetric theories. In contrast to these issues, we are focusing here mainly on methods to calculate properties of particular (quasifree) fermion states. For this purpose, we need to consider the appropriate algebraic structure in a representation independent manner. As it turns out, the Grassmann algebra of canonical anticommutation relations (GAR) algebra is the right concept to be used here. A direct way of explaining the GAR algebra for the finite dimensional case is given in terms of standard creation and annihilation operators of fermionic modes: We consider $2n+k$ fermionic modes and take all the creation operators $c_1^*, c_2^*, \ldots, c_{2n}^*, c_{2n+1}^*, \ldots, c_{2n+k}^*$ and the last k annihilation operators $c_{2n+1}, \ldots, c_{2n+k}$. Then we build the subalgebra generated by these operators, being represented on the antisymmetic Fock space over $\mathbb{C}^{2n+k}$. Obviously, this algebra is not closed under the usual adjoint since we only take the creation operators for the first $2n$ modes. In fact, the first $2n$ annihilation operators generate the exterior algebra or *Grassmann algebra* over $\mathbb{C}^{2n}$. The last k modes are identified with the usual Fermion algebra. In order to implement a *complex conjugation* for Grassmann variables, we introduce a "new" adjoint $\star$ on the GAR algebra. For a generator c_i that belongs to the first $2n$ modes, it is defined by $c_i^\star := c_{2n-i+1}$. For a generator that belongs to the last k modes we just take $c_j^\star := c_j^*$ the usual adjoint. It follows directly from this construction that the defining representation on antisymmetric Fock space is not a *-representation, i.e., it does not preserve the adjoint.

The GAR algebra consists of a *fermionic part*, that is, generated by Fermi field operators that are linear combinations of creation and annihilation operators,

$$B(f) = \sum_{i=2n+1}^{2n+k} f_+^i c_i^* + f_-^i c_i, \tag{1}$$

that fulfill the anticommutation relations

$$\{B(f), B(h)\} = \sum_{i=2n+1}^{2n+k} (f_+^i h_-^i + f_-^i h_+^i)\mathbb{1}. \tag{2}$$

The fermionic part of the GAR algebra always corresponds to the underlying fermion system one wishes to investigate.

The *Grassmann part* of the GAR algebra is generated by the first $2n$ modes $c_1^*, \ldots, c_{2n}^*$. Obviously, the GAR algebra possesses a natural $\mathbb{Z}_2$-grading by looking at the subspaces of *even* and *odd* operators. Here, we call a GAR operator to be even (odd) if it is a complex linear combination of even (odd) products of operators $c_1^*, c_2^*, \ldots, c_{2n}^*, c_{2n+1}^*, \ldots, c_{2n+k}^*, c_{2n+1}, \ldots c_{2n+k}$.

One of the main ideas behind introducing the GAR algebra is to build up an appropriate extension of the fermion algebra in which the anticommutation relations can be written in terms of commutation relations by "substituting" the complex linear combinations of creation and annihilation operators by linear combinations with "anticommuting" variables as coefficients. These anticommuting variables, called *Grassmann variables*, are linear combinations of odd products of the creation operators $c_1^*, c_2^*, \ldots, c_{2n}^*$. Suppose that $\xi = (\xi_+^1, \ldots, \xi_+^k, \xi_-^1, \ldots, \xi_-^k)$ is a vector of $2k$ odd operators from the Grassmann part. Then the linear combination,

$$\Phi(\xi) = \sum_{i=1}^{k} \xi_+^i c_{i+2n}^* + \xi_-^i c_{i+2n}, \tag{3}$$

is a well-defined operator inside the GAR algebra. These operators are the *Grassmann–Bose fields*. A straightforward calculation shows that the commutation relation

$$[\Phi(\xi),\Phi(\eta)] = -\sum_{i=1}^{k} \xi_+^i \eta_-^i - \eta_+^i \xi_-^i = \sigma(\xi,\eta) \tag{4}$$

is fulfilled. The bilinear form σ can be interpreted a symplectic form with values in the Grassmann algebra. Therefore, the Grassmann–Bose field operators $\Phi(\xi)$ fulfill a kind of "canonical commutation relations." Here, one has to be aware of the fact that the right hand side is not a complex multiple of the identity but an operator that belongs to the center of the GAR algebra. The *Grassmann–Weyl operators* are given by the exponential $\mathbf{w}(\xi)=\exp(\Phi(\xi))$ and they fulfill the Weyl relations,

$$\mathbf{w}(\xi+\eta) = e^{(1/2)\sigma(\xi,\eta)}\mathbf{w}(\xi)\mathbf{w}(\eta). \tag{5}$$

Our fermionic analog of a "quantum harmonic analysis on phase space" (see Ref. 23) is based on these structures.

Our paper is outlined into tree main sections, where Sec. II is for presenting basic definitions and main results as precise as possible on one hand, and as less technical as possible on the other hand. The two following sections are more technical in order to explain the mathematics we are using. We also introduce here all mathematical concepts that are needed for our analysis. To make this part more readable, we postpone here the discussion of technical details. We also present here the main results concerning a harmonic analysis on fermionic phase space and we present its application to calculate the fidelity between quasifree fermion states. Section III is dedicated to a detailed discussion on the mathematical structure of the GAR algebra. Statements which have been claimed in the previous section are proven here. Finally, Sec. IV, we review the concept of Grassmann integration adopted to our analysis.

II. BASIC DEFINITIONS AND MAIN RESULTS

A. Preliminaries

There are two basic notions that will play an essential role within this paper. First, the concept of a Banach *-algebra and the concept of a C*-algebra. We briefly recall these concepts here in order to give a precise formulation of the mathematical structure we are going to use.

- A *Banach algebra* is a associative algebra and a Banach space with a norm $\|\cdot\|$, such that the relation $\|AB\| \le \|A\|\|B\|$ is fulfilled.
- *An adjoint* is a continuous antilinear involution * which fulfills $(cA)^\star = \bar{c}A^\star$ for an operator A and a complex number c and the order in a product is reversed $(AB)^\star = B^\star A^\star$.
- A *Banach* *-algebra is a Banach algebra with an adjoint.
- A C*-algebra is a Banach *-algebra, where the adjoint * fulfills the C*-condition $\|A^*A\| = \|A\|^2$. Vice versa, an adjoint * on a Banach algebra is called a C* *-adjoint* if the C*-condition holds.

In view of treating superspaces and supersymmetry Banach *-algebras has been used, for instance, in Ref. 13.

Later on it will be an important issue to distinguish the adjoint of a C*-algebra with the adjoint of a generic *-algebra. The convention we will use here is to denote the C*-adjoint of an operator A by A^* and the adjoint in a generic *-algebra by $A^\star$.

In order to introduce the GAR algebra in a most general manner, we briefly recall here *Araki's self-dual CAR algebra*.[1] Let H be a separable Hilbert space with a complex conjugation J. Note that H may be infinite dimensional. Then there exists a unique C*-algebra $\mathcal{F}(H,J)$, the so-called *algebra of canonical anti-commutation relations (CAR)*, that is, generated by operators $B(f)$ with $f \in H$, such that $f \mapsto B(f)$ is a complex linear map, the C*-adjoint of $B(f)$ is given by $B(f)^* = B(Jf)$, and the anticommutator fulfills the relation $\{B(f),B(h)\}=\langle Jf,h\rangle \mathbb{1}$.

Remark 1: The main advantage of using Araki's self-dual description of the CAR algebra is that it is independent of the chosen representation. To obtain a representation in terms of creation and annihilation operators, Eq. (1) is an example for the finite dimensional Hilbert space $H=\mathbb{C}^{2k}$,

where the complex conjugation is given by $J(f_-^1, \ldots, f_-^k, f_+^1, \ldots, f_+^k) = (\bar{f}_+^1, \ldots, \bar{f}_+^k, \bar{f}_-^1, \ldots, \bar{f}_-^k)$.

B. The GAR algebra

The basic ingredients for construction the GAR algebra are a separable Hilbert space H, a complex conjugation J, and a projection Q that commutes with J. We consider the Hilbert space $H_Q := H \oplus Q^\perp H$ with the complex conjugation $J_Q := R_Q(J \oplus Q^\perp J)$, where $Q^\perp = \mathbb{1} - Q$ is the projection onto the orthogonal complement of QH and the reflection R_Q is defined according to $R_Q(f \oplus h) = (Qf + h) \oplus Q^\perp f$ for $f \in H$, $h \in Q^\perp H$. We associate with the triple (H, Q, J) the CAR algebra $\mathcal{E}(H, Q; J) := \mathcal{F}(H_Q, J_Q)$ in view of the following definition.

Definition 2:

(1) The *GAR algebra* $\mathcal{G}(H, Q; J)$ associated with the triple (H, Q, J) is the norm-closed subalgebra of the CAR algebra $\mathcal{E}(H, Q; J)$, that is, generated by the *Grassmann–Fermi field operators* $G(f) := B(f \oplus 0)$, $f \in H$.

(2) The *open core* $\mathring{\mathcal{G}}(H, Q; J)$ of the GAR algebra is the subalgebra that consists of finite sums of finite products of Grassmann–Fermi field operators.

(3) The CAR algebra $\mathcal{E}(H, Q; J)$ is called the *enveloping CAR algebra* of $\mathcal{G}(H, Q; J)$.

(4) The *fermionic part* of the GAR algebra is the norm-closed subalgebra of $\mathcal{E}(H, Q; J)$, that is, generated by the Grassmann–Fermi fields $G(Qf)$, $f \in QH$.

(5) The *Grassmann part* of the GAR algebra is the norm-closed subalgebra of $\mathcal{E}(H, Q; J)$, that is, generated by the Grassmann–Fermi fields $G(Q^\perp f)$, $f \in Q^\perp H$.

As a consequence of this definition, the GAR algebra $\mathcal{G}(H, Q; J)$ is a Banach algebra and the Grassmann–Fermi field operators fulfill the anticommutation relations

$$\{G(f), G(h)\} = \langle Jf, Qh \rangle \mathbb{1} \tag{6}$$

for $f, h \in H$. This can be verified by calculating the anticommutator $\{G(f), G(h)\} = \{B(f \oplus 0), B(h \oplus 0)\} = \langle J_Q(f \oplus 0), h \oplus 0 \rangle \mathbb{1} = \langle R_Q(Jf \oplus 0), h \oplus 0 \rangle \mathbb{1} = \langle JQf, h \rangle \mathbb{1} = \langle Jf, Qh \rangle \mathbb{1}$. From this calculation, it also follows that the fermionic part of $\mathcal{G}(H, Q; J)$ coincides with the CAR subalgebra $\mathcal{F}(QH, QJ) \subset \mathcal{E}(H, Q; J)$.

As already mentioned, inside the GAR algebra we can build linear combinations of fermion operators with coefficients in the Grassmann algebra which yields the possibility of building fields with "canonical commutation relations" inside the GAR algebra. For this purpose, it is also important to have the concept of an adjoint. The problem is here, that the C*-adjoint in the enveloping CAR algebra cannot be used, since the GAR algebra is not closed under this operation. Namely, for a generator $G(Q^\perp f)$ the C*-adjoint is given by $G(Q^\perp f)^* = B(Q^\perp f \oplus 0)^* = B(0 \oplus JQ^\perp f) \notin \mathcal{G}(H, Q; J)$. Only the fermionic part is stable under the C*-adjoint. We shall see (Proposition 13) that there exists an adjoint $\star : \mathcal{G}(H, Q; J) \to \mathcal{G}(H, Q; J)$, such that the GAR algebra becomes a Banach *-algebra and that the adjoint $\star$ coincides with the C*-adjoint on the fermionic part. Moreover, the adjoint $\star$ is uniquely determined by the relation $G(f)^\star = G(Jf)$ for $f \in H$.

Remark 3: The Grassmann algebra can be regarded as a special case of the GAR algebra, where the projection Q is chosen to be zero. To be more precise, the *Grassmann algebra* $\Lambda(H, J)$ over the pair H, J is defined as the GAR algebra $\Lambda(H, J) := \mathcal{G}(H, 0; J)$. On the other hand, the Grassmann algebra can be constructed from the antisymmetric tensor algebra $\mathring{\Lambda}(H, J) := \oplus_{k \in \mathbb{N}} \wedge^k H$ over the Hilbert space H. As a linear space, the antisymmetric tensor algebra $\mathring{\Lambda}(H, J)$ is a dense subspace of the antisymmetric Fock space $F_-(H)$ over H. Thus, we can equip $\mathring{\Lambda}(H, J)$ with a scalar product $\langle, \rangle$. This gives rise to a further norm on $\mathring{\Lambda}(H, J)$, that is, given by $\|\|\lambda\|\| := \sqrt{\langle \lambda, \lambda \rangle}$. As we will see later, this norm is *not* a Banach algebra norm, but it is continuous with respect to the Banach algebra norm $\|\cdot\|$, i.e., $\|\|A\|\| \leq \|A\|$.

Remark 4: By construction, the GAR algebra is isomorphic to the twisted (graded) tensor product (see Refs. 6 and 7, for this notion) of the fermionic part and the Grassmann part. Following the analysis of Ref. 14 the Grassmann algebra can be regarded as the classical limit of a field

of CAR algebras. Analogously, the GAR algebra can be viewed as a *partial* classical limit of a field of CAR algebras. The basic idea behind their work is to introduce for a Hilbert space H, a complex conjugation J, and a positive number $\hbar > 0$ the modified CAR algebra $\mathcal{F}(H_\hbar, J)$, where $H_\hbar$ is the Hilbert space with the scaled scalar product $\langle f, h \rangle_\hbar = \hbar \cdot \langle f, h \rangle$. Roughly, in the classical limit of the field of CAR algebras $(\mathcal{F}(H_\hbar, J), \hbar > 0)$ becomes the Grassmann algebra $\Lambda(H, J)$ which is based on the behavior anticommutator relations $\lim_{\hbar \to 0} \{ B_\hbar(f), B_\hbar(h) \} = \lim_{\hbar \to 0} \hbar \langle f, h \rangle \mathbb{1} = 0$.

To view the GAR algebra as a partial classical limit we consider the Hilbert space $H_{\hbar, Q}$ with the partially scaled scalar product $\langle f, h \rangle_\hbar := \langle f, (Q + \hbar Q^\perp) h \rangle$. The CAR algebra $\mathcal{F}(H_{\hbar, Q}, J)$ is now isomorphic to the twisted tensor product of $\mathcal{F}(QH, QJ)$ and $\mathcal{F}((Q^\perp H)_\hbar, Q^\perp J)$. Keeping in mind that the classical limit of the field of CAR algebras $(\mathcal{F}((Q^\perp H)_\hbar, Q^\perp J), \hbar > 0)$ is the Grassmann algebra $\Lambda(Q^\perp H, Q^\perp J)$, the GAR algebra is the partial classical limit of the CAR algebra $\mathcal{F}(H_{\hbar, Q}, J) \to \mathcal{G}(H, Q; J)$.

Remark 5: For the case that the projection Q is a projection of even and finite dimension $2n$, there is a further simple characterization of the GAR algebra. Namely, the GAR algebra can also be seen as a matrix algebra with Grassmann valued entries. To verify this, we use the fact (as in the previous remark) that the GAR algebra $\mathcal{G}(H, Q; J)$ is the twisted tensor product of the CAR algebra $\mathcal{F}(QH, QJ)$ and the Grassmann algebra $\Lambda(Q^\perp H, Q^\perp J)$. Recall that the isomorphism is given by $G(f) \mapsto G(Qf) \otimes \mathbb{1} + \Theta \otimes G(Q^\perp f)$, where Θ is the reflection fulfilling $\Theta G(Qf) = -G(Qf) \Theta$. Moreover, the fermionic part is isomorphic to the algebra $\mathrm{M}_{2^{2n}}(\mathbb{C})$ of complex $2^{2n} \times 2^{2n}$ matrices. By choosing a matrix basis E_{ij}, $i, j = 1, \dots, 2^{2n}$, each operator in the GAR algebra can uniquely be expanded as $A = \Sigma_{ij} E_{ij} A_{ij}$, where the operators A_{ij} belong to the Grassmann part. Thus, the desired isomorphism identifies the operator A with the matrix (A_{ij}) belonging to the algebra $\mathrm{M}_{2^{2n}}(\Lambda(Q^\perp H, Q^\perp J))$ of $2^{2n} \times 2^{2n}$ matrices with entries in the Grassmann part.

C. States

The GAR algebra possesses a natural convex cone of positive elements. First, the set of positive linear functionals consists of all linear functionals $\omega : \mathcal{G}(H, Q; J) \to \mathbb{C}$ with $\omega(A^\star A) \geq 0$. Second, the positive cone $\mathcal{G}(H, Q; J)_+$ consists of all operators that have positive expectation values for all positive functionals.

In order to analyze the positivity of operators, we introduce the norm-closed two-sided ideal $\mathcal{I}(H, Q; J)$, that is, generated by the *self-adjoint nilpotent operators* in the Grassmann part $\Lambda(Q^\perp H, Q^\perp J)$. Recall that an operator Z is nilpotent if there exists $n \in \mathbb{N}$ with $Z^n = 0$. It can be shown (see Proposition 15) that to each positive functional ω on the GAR algebra $\mathcal{G}(H, Q; J)$, there exists a unique positive functional ω' on the fermionic part $\mathcal{F}(QH, QJ)$, such that

$$\omega(A + Z) = \omega'(A), \tag{7}$$

where A is an operator in the fermionic part $\mathcal{F}(QH, QJ)$ and Z belongs to the ideal $\mathcal{I}(H, Q; J)$. We refer the reader to Sec. III C for a more detailed discussion. This shows that the positive functional on the GAR algebra is in one-to-one correspondence with the positive functionals on the fermionic part.

Instead of considering complex valued functionals, the appropriate concept is as it turns out later, to consider functionals from the GAR algebra into its Grassmann part. The GAR algebra is equipped with a natural right module structure over the Grassmann part via multiplication from the right. For our purpose, the appropriate method is to extend a state on the fermionic part as a right module homomorphism. For a linear functional ω on the CAR algebra $\mathcal{F}(QH, QJ)$, we are seeking for a linear map $\boldsymbol{\omega} : \mathcal{G}(H, Q; J) \to \Lambda(Q^\perp H, Q^\perp J)$ which fulfills the condition

$$\boldsymbol{\omega}(A\lambda) = \boldsymbol{\omega}(A)\lambda. \tag{8}$$

We call $\boldsymbol{\omega}$ the *G-extension* of ω to the GAR algebra. To obtain the G-extension of a state on the fermionic part, we use the fact that the developing CAR algebra $\mathcal{E}(H, Q; J)$ can be identified with

the twisted (graded) tensor product of the fermionic part and the enveloping CAR algebra of the Grassmann part,

$$\mathcal{E}(H,Q;J) = \mathcal{F}(QH,QJ)\widetilde{\otimes}\mathcal{E}, \tag{9}$$

where $\mathcal{E}$ denotes the enveloping CAR algebra of the Grassman part. For a vector $f\oplus h\in H\oplus Q^{\perp}H$, the corresponding Fermi field operator is identified with the tensor product by

$$B(f\oplus h) = B(Qf)\otimes\mathbb{1} + \Theta_Q\otimes B(Q^{\perp}f\oplus h), \tag{10}$$

where Θ_Q is the reflection that implements the parity automorphism $\Theta_Q B(Qf)\Theta_Q = -B(Qf)$. As a Banach space, the enveloping CAR algebra is identified with a tensor product of two C*-algebras. With respect to the positivity structure of this tensor product, the linear map $\omega\otimes\mathrm{id}_{\mathcal{E}}$ is completely positive. Therefore, it is bounded as a map between Banach spaces. The G-extension is now given by the restriction to the GAR algebra,

$$\boldsymbol{\omega} := \omega\otimes\mathrm{id}_{\mathcal{E}}\big|_{\mathcal{G}(H,Q;J)}, \tag{11}$$

which becomes a bounded map from the GAR algebra into its Grassmann part. By construction, the condition (8) is fulfilled.

Remark 6: For our further analysis, the essential property of the G-extension is to be a right module homomorphism (8), whereas positivity with respect to the adjoint of the GAR is not essential. Anyway, from the above construction, we cannot conclude directly that the G-extension is positive since the GAR algebra has a different positivity structure than the tensor product $\mathcal{F}(QH,QJ)\otimes\mathcal{E}$.

D. Anticommutative phase space

We have introduced the GAR algebra in terms of the Grassmann–Fermi field operators $G(f)$. In this section we introduce a different family of field operators, called Grassmann–Bose fields, that also generate, together with the unit operator, the GAR algebra. It turns out that these fields fulfill a graded version of the canonical commutation relations.

We introduce the *anticommutative phase space* as the tensor product $\mathcal{R}(H,Q;J) := QH\otimes\Lambda(Q^{\perp}H,Q^{\perp}J)$. We are considering here a tensor product of a Hilbert space and a Grassmann algebra closed with respect to the projective cross norm $\|\cdot\|_{\gamma}$ (see, for instance, Ref. 20). Since the projective cross norm is the largest among all cross norms, it follows that the anticommutative phase space $\mathcal{R}(H,Q;J)$ can be identified with a linear subspace of the Grassmann algebra $\Lambda(H,J)$ by the continuous embedding which identifies the tensor product $f\otimes\lambda$ with the operator $\Lambda(f)\lambda$. In order to express the (anti)commutation relations for the Grassmann–Bose fields, we equip the anticommutative phase space $\mathcal{R}(H,Q;J)$ with a continuous Grassmann valued inner product. This *rigging map* $\langle\cdot,\cdot\rangle_Q$ is determined on pure tensor products by $\langle f\otimes\lambda, h\otimes\mu\rangle_Q := \langle f,h\rangle\lambda^{\star}\mu$.

Remark 7: If Q is a projection of finite and even rank $2n$, then the anticommutative phase space is simply isomorphic to the $2n$-fold Cartesian product of the Grassmann algebra. This can be seen by choosing a real orthonormal basis $(e^i)_{i=1,\ldots,2n}$ of QH. Each phase space vector ξ can be uniquely expanded as $\xi = \Sigma_i e^i\otimes\xi_i$, where ξ_i is an operator from the Grassmann part. Thus, an isomorphism between $\mathcal{R}(H,Q;J)$ and $\Lambda(Q^{\perp}H,Q^{\perp}J)^{2n}$ is given by $\xi\mapsto(\xi_1,\ldots,\xi_{2n})$.

The Grassmann–Bose field Φ is a right module homomorphism that associates with each phase space vector ξ an operator $\Phi(\xi)$ in the GAR algebra. This map is determined on pure tensor products $\xi = f\otimes\lambda$ according to

$$\Phi(f\otimes\lambda) := G(f)\lambda \tag{12}$$

where $f\in QH$ and λ belonging to the Grassmann part. Note that the inequality $\|\Phi(\xi)\|\leq\|\xi\|_{\gamma}$ holds which implies that the map Φ is continuous and can uniquely be extended to the full anticommutative phase space.

The $\mathbb{Z}_2$-grading of the Grassmann part induces a direct sum decomposition of the Banach space $\mathcal{R}(H,Q;J) = \mathcal{R}(H,Q;J)_0 \oplus \mathcal{R}(H,Q;J)_1$ with $\mathcal{R}(H,Q;J)_q := QH \otimes \Lambda(Q^\perp H, Q^\perp J)_{q+1}$. Moreover, we introduce a complex conjugation according to $(f \otimes \lambda)^\star := (-1)^{q+1} Jf \otimes \lambda^\star$ with $\lambda \in \Lambda(Q^\perp H, Q^\perp J)_q$. The grading and the complex conjugation are compatible with the identification of $\mathcal{R}(H,Q;J)$ as a closed linear subspace of the Grassmann algebra $\Lambda(H,J)$. Here the tensor product $f \otimes \lambda$ is just identified with the operator $\Lambda(f)\lambda$. The complex conjugation and the grading in $\mathcal{R}(H,Q;J)$ are nothing else but the adjoint and the grading within the ambient Grassmann algebra. The GAR relations can be expressed in terms of graded commutators. Recall that the graded commutator of $[A,B]_\mathrm{g}$ is given by the commutator $[A,B]$ if A or B are even and by the anticommutator $\{A,B\}$ if both A and B are odd.

It follows from the construction of the Grassmann–Bose fields that for a pair of phase space vectors $\xi, \eta \in \mathcal{R}(H,Q;J)$, the graded commutator fulfills

$$[\Phi(\xi), \Phi(\eta)]_\mathrm{g} = \langle \xi^\star, \eta \rangle_Q. \tag{13}$$

Moreover, the adjoint fulfills the identity $\Phi(\xi)^\star = \Phi(\xi^\star)$. For the particular case that ξ, η are even elements in $\mathcal{R}(H,Q;J)$ the Grassmann–Bose field fulfills the canonical commutation relations,

$$[\Phi(\xi), \Phi(\eta)] = \langle \xi^\star, \eta \rangle_Q. \tag{14}$$

Note that the restriction of the rigging map to the even subspace is antisymmetric, i.e., $\langle \xi^\star, \eta \rangle_Q = -\langle \eta^\star, \xi \rangle_Q$ for ξ, η even. As for the usual formulation of the canonical commutation relation, the commutator belongs to the center of the GAR algebra. The main difference is here that the Grassmann–Bose field operators are bounded in norm. This is no contradiction, since we are dealing here with Banach *-algebras (rather than C*-algebras).

For an even $\xi \in \mathcal{R}(H,Q;J)_0$ the exponential $\mathbf{w}(\xi) := \exp(\Phi(\xi))$ of the Grassmann–Bose field operator is well defined. We call $\mathbf{w}(\xi)$ the *Grassmann–Weyl operator* for ξ. Since field operators $\Phi(\xi), \Phi(\eta)$ are even, the Grassmann–Weyl operators fulfill the relations

$$\mathbf{w}(\xi + \eta) = \mathrm{e}^{(1/2)\langle \xi^\star, \eta \rangle_Q} \mathbf{w}(\xi) \mathbf{w}(\eta). \tag{15}$$

As for ordinary Weyl operators, the map $\xi \mapsto \mathbf{w}(\xi)$ is a projective representation of the additive group $\mathcal{R}(H,Q;J)_0$, where the factor system belongs to the center of the GAR algebra.

Obviously the Grassmann–Weyl operator $\mathbf{w}(\xi)$ is unitary only if $\xi = \xi^\star$ is self-adjoint. If we restrict the Grassmann–Weyl system to self-adjoint phase space vectors, then the value of the rigging map $\langle \xi^\star, \eta \rangle_Q = \langle \xi, \eta \rangle_Q = -\langle \xi, \eta \rangle_Q^\star$ is anti-self-adjoint and the exponential in the Grassmann–Weyl relation is also unitary. Note that $\mathcal{R}(H,Q;J)_0$ should be seen as a "complexified" anticommutative phase space and the Grassmann–Weyl operators are directly constructed as a kind of analytic continuation from the real part.

E. Toward a harmonic analysis on anticommutative phase space

The concept of anticommutative phase space can be used to perform a kind of "harmonic analysis" that is analogous to the analysis of the bosonic case.[23] As for the case of ordinary symplectic vector spaces,[23] we introduce here the analogous concept of *convolution* and *Fourier transform*. This requires to "integrate" over antisymmetric phase space. Here the Brezin–Grassmann integration (see, for instance, Refs. 7, 21, and 22) turns out to be the appropriate notion which we recall here. Before we continue our discussion, we mention the following.

In order to perform integration with respect to Grassmann variables, we have to consider the algebra of functions that can be integrated. These functions are appropriate polynomials of Grassmann variables $\xi \in \mathcal{R}(H,Q;J)_0$ with values in a right module over the ring $\Lambda(Q^\perp H, Q^\perp J)$. If the underlying ring structure is clear from the context we just briefly say "right module." We assume here that the projection Q has finite even rank $\dim(Q) = 2n$, which corresponds to an integration over a finite dimensional space.

We need to integrate functions with values in a right module $\mathcal{E}$ that admit a *polynomial*

representation in terms of Grassmann variables. However, this representation has some ambiguities which causes some problems in defining the Grassmann integral. A polynomial representation is obtained from a real orthonormal basis $(e^i)_{i \in N}$ of QH, that is, indexed by the ordered set $N = \{1, \ldots, \dim(Q)\}$. Any vector $\xi \in \mathcal{R}(H, Q; J)_0$ can be expanded in this basis as $\xi = \Sigma_i \Lambda^i \xi_i$ with $\xi_i \in \Lambda(Q^\perp H, Q^\perp J)_1$ and $\Lambda^i := \Lambda(e^i)$. With respect to this basis, the polynomial representation of a G-holomorphic function F is given by

$$F(\xi) = \sum_{I \subset N} F^I \xi_I. \tag{16}$$

Here the coefficients F^I are contained in the right module $\mathcal{E}$. The monomial ξ_I which is associated with an ordered subset $I = \{i_1 < i_2 < \cdots < i_k\} \subset N$ is given by $\xi_I := \xi_{i_1} \cdots \xi_{i_k}$.

For a given polynomial representation, the Brezin–Grassmann integral of F over a form of highest degree (2n-form) $v = v_N e^1 \wedge \cdots \wedge e^{2n}$ in $\Lambda(QH, QJ)$ is defined according to

$$\int_Q v(\xi) F(\xi) = v_N F^N. \tag{17}$$

The problem with this definition is that, in general, the relation

$$\sum_{I \subset N} F^I \xi_I = 0 \tag{18}$$

is nontrivial in the sense that (18) holds for nonzero coefficients F^I. This means that the polynomial representation for a given basis is not unique and the integral may not be well defined. However, for some right modules $\mathcal{E}$ the highest coefficient F^N is unique. This is the case if $\mathcal{E}$ fulfills the following condition: If $X\lambda_1 \cdots \lambda_{2n} = 0$ for all odd Grassmann operators $\lambda_1, \ldots, \lambda_{2n}$, then $X = 0$. This property holds for $\mathcal{E} = \mathcal{G}(H, Q; J)$ and $\mathcal{E} = \Lambda(Q^\perp H, Q^\perp J)$, if the complementary projection $Q^\perp$ is infinite dimensional. A more detailed analysis of this issue is postponed to Sec. IV A. In what follows, we assume that Q is a projection of finite dimension $2n$ and its complement $Q^\perp$ is infinite dimensional.

We introduce the Banach space $\mathrm{Hom}(H, Q, J)$ of bounded right module homomorphisms from the GAR algebra into its Grassmann part. As already mentioned, the space $\mathcal{R}(H, Q; J)_0$ can be regarded as anticommutative phase space with a complex structure that is given by the adjoint $\star$ where the rigging map $\langle, \rangle_Q$ induces a Grassmann valued symplectic form on $\mathcal{R}(H, Q; J)_0$. The phase space translations act by automorphisms on the GAR algebra by the adjoint action of the Grassmann–Weyl operators,

$$\alpha_\xi(A) = \mathbf{w}(-\xi) A \mathbf{w}(\xi). \tag{19}$$

Note that α_ξ is a *-automorphism if and only if the operator ξ is self-adjoint, i.e., it belongs to the real part of $\mathcal{R}(H, Q; J)_0$.

By fixing a self-adjoint and normalized form v of highest degree with respect to Q, we are now prepared to define the following convolutions.

(1) The convolution of two G-holomorphic functions is a G-holomorphic function that is given by

$$\mathcal{O}(H, Q, J) \times \mathcal{O}(H, Q, J) \ni (f_1, f_2) \rightarrow f_1 * f_2 \in \mathcal{O}(H, Q, J),$$

$$(f_1 * f_2)(\xi) := \int_Q v(\eta) f_1(\eta) f_2(\xi - \eta). \tag{20}$$

(2) The convolution of an operator in the GAR algebra with a G-holomorphic function is the operator in the GAR algebra that is given by

$$\mathcal{G}(H,Q;J) \times \mathcal{O}(H,Q,J) \ni (A,f) \rightarrow A * f \in \mathcal{G}(H,Q;J),$$

$$A * f := \int_Q v(\eta)\alpha_{-\eta}(A)f(\eta). \tag{21}$$

(3) The convolution of a right module homomorphism with an operator in the GAR algebra is a G-holomorphic function that is given by

$$\mathrm{Hom}(H,Q,J) \times \mathcal{G}(H,Q;J) \ni (\varphi,A) \rightarrow \varphi * A \in \mathcal{O}(H,Q,J),$$

$$(\varphi * A)(\xi) := \varphi(\alpha_\xi(A)). \tag{22}$$

As for the convolution, we also have three different cases for applying the *Fourier transform*.

(1) The Fourier transform maps an operator of the reduced GAR algebra to a G-holomorphic function according to

$$\mathcal{F}:\mathcal{G}(H,Q;J) \rightarrow \mathcal{O}(H,Q,J),$$

$$(\mathcal{F}A)(\xi) := \mathbf{w}(-\xi)\int_Q v(\eta)\alpha_{-\eta}(A)e^{\langle\xi^\star,\eta\rangle_Q}. \tag{23}$$

Up to now, the function $\mathcal{F}A$ can have values in the GAR algebra, but we will show that the range of the function is indeed fully contained in the Grassmann part.

(2) The Fourier transform maps a G-holomorphic function to a G-holomorphic function according to

$$\mathcal{F}:\mathcal{O}(H,Q,J) \rightarrow \mathcal{O}(H,Q,J),$$

$$(\mathcal{F}f)(\xi) := \int_Q v(\eta)f(\eta)e^{\langle\xi^\star,\eta\rangle_Q}. \tag{24}$$

(3) Finally, the Fourier transform of a right module homomorphism is the G-holomorphic function which is given by its expectation values of the Grassmann–Weyl operators in the GAR algebra,

$$\mathcal{F}:\mathrm{Hom}(H,Q,J) \rightarrow \mathcal{O}(H,Q,J),$$

$$(\mathcal{F}\varphi)(\xi) = \varphi(\mathbf{w}(\xi)). \tag{25}$$

F. Main results

A useful fact is that all operators of the GAR algebra $\mathcal{G}(H,Q;J)$ can be represented in terms of Grassmann integrals as stated by the following theorem.

Theorem 8: *The Fourier transform $\mathcal{F}$ maps each operator $A \in \mathcal{G}(H,Q;J)$ of the GAR algebra to a G-holomorphic function in $\mathcal{O}(H,Q,J)$, such that the identity*

$$A = \int_Q v(\xi)\mathbf{w}(\xi)(\mathcal{F}A)(\xi) \tag{26}$$

holds.

The Fourier transform maps the convolution of objects into their product of Fourier transforms as stated by the following theorem.

Theorem 9: *Let f,f' be G-holomorphic functions, let A an operator of the GAR algebra, and*

let φ be a right module homomorphism. Then the identities

$$\mathcal{F}(f * f') = \mathcal{F}f\mathcal{F}f',$$

$$\mathcal{F}(\varphi * A) = \mathcal{F}\varphi\mathcal{F}A,$$

$$\mathcal{F}(A * f) = \mathcal{F}A\mathcal{F}f \tag{27}$$

are valid.

The integral representation for operators in the GAR algebra can be used to calculate the expectation values of a right module homomorphism on the GAR algebra in terms of the expectation values of Grassmann–Weyl operators. According to the discussion above, the identities

$$\boldsymbol{\omega}(A) = \int_Q v\mathcal{F}\boldsymbol{\omega}\mathcal{F}A = \int_Q v\mathcal{F}(\boldsymbol{\omega} * A) \tag{28}$$

are valid.

Corollary 10: Let $\boldsymbol{\omega}, \varphi$ bounded right module homomorphisms. If $\mathcal{F}\varphi$ is a divisor of $\mathcal{F}\boldsymbol{\omega}$ within the ring of G-homomorphic functions, then the Radon–Nikodym-type relation,

$$\boldsymbol{\omega}(A) = \int_Q v\frac{\mathcal{F}\boldsymbol{\omega}}{\mathcal{F}\varphi}\mathcal{F}(\varphi * A), \tag{29}$$

is valid for all operators A of the GAR algebra.

For the presentation of our next results, we need to recall the definition of the *quasifree fermion states* as well as the definition of the *Pfaffian* of a real antisymmetric matrix, as well as how to calculate Gaussian Grassmann integrals.

Quasifree states. Each quasifree state on the fermionic part $\mathcal{F}(QH, QJ)$ is in one-to-one correspondence with its covariance matrix. This is a linear operator S on QH with $0 < S \le \mathbb{1}$ and it has to fulfill the constraint $S + JSJ = \mathbb{1}$. The expectation values of the state ω_S are related to its covariance matrix S by the following condition on the two-point correlation function:

$$\omega_S(B(f)B(h)) = \langle Jf, Sh \rangle \tag{30}$$

with $f, h \in QH$. All higher correlation functions can be expressed in terms of sums of products of two-point functions according to Wick theorem, where only the expectation values of an even product of Fermi field operators are nonvanishing. It is well known that a quasifree state is pure if and only if its covariance matrix $S = P$ is a projection, called *basis projection*.

Pfaffian. It is well known that Gaussian Grassmann integrals can be expressed in terms of the Pfaffian of the corresponding covariance matrix. The nth power of a two-form $a \in \Lambda(QH, QJ)$ is a $2n$-form in $\Lambda(QH, QJ)$ and therefore proportional to any other $2n$-form. By fixing a selfadjoint normalized $2n$-form v, i.e., $v = v^\star$ and $\langle v, v \rangle = 1$, there exists a complex number $\mathrm{Pf}_{[v]}(a)$, called the *Pfaffian* of a that is uniquely determined by

$$(n!)^{-1}a^n = (n!)^{-1}\langle v, a^n \rangle v = \mathrm{Pf}_{[v]}(a)v. \tag{31}$$

Now, let A be a linear operator on QH, then there exists a unique two-form a, such that $\langle a^\star, f \wedge h \rangle = \langle Jf, Ah \rangle$, where the two-form a only depends on the J-antisymmetric part $(A - JA^*J)/2$ of the operator A. Note that the J-antisymmetry is related to *the transpose $A \mapsto JA^*J$*. The Pfaffian of a J-antisymmetric operator A is now defined as

$$\mathrm{Pf}_{[v,J]}(A) := \mathrm{Pf}_{[v]}(a), \tag{32}$$

where a is the two-form fulfilling the identity $\langle a^\star, f \wedge h \rangle = \langle Jf, Ah \rangle$. If we restrict to the real subspace in QH that is given by $Jf = f$, we obtain therefore the standard definition of the Pfaffian for

a real antisymmetric operator. In particular, the determinant of an antisymmetric operator with respect to the $2n$-form v is given by $\det(A)v = \Gamma(A)v$, where $\Gamma(A)$ denotes the second quantized operator of A on the antisymmetric Fock space over QH that is given by $\Gamma(A)(f_1 \wedge \cdots \wedge f_k) = Af_1 \wedge \cdots \wedge Af_k$. We recall here the well known identity

$$\det(A) = \mathrm{Pf}_{[v,J]}(A)^2. \tag{33}$$

Note that due to the condition $A = -JA^*J$ the left hand side implicitly also depends on the complex conjugation J. However, whereas the determinant can be defined for any linear operator on QH, the Pfaffian is only defined on the *real* linear subspace of J-antisymmetic operators.

   Theorem 11: *For each covariance operator S on QH the Fourier transform of the G-extension $\boldsymbol{\omega}_S$ of the quasifree state ω_S is given by*

$$\mathcal{F}\boldsymbol{\omega}_S(\xi) = \mathrm{e}^{-(1/2)\langle \xi^\star, S\xi\rangle_Q}. \tag{34}$$

Moreover, let P be a basis projection and let E_P be the support projection of the pure quasifree state ω_P. Then for a normalized self-adjoint form v of highest degree with respect to Q the identity

$$\mathcal{F}(\boldsymbol{\omega}_P * E_P)(\xi) = \epsilon_{[v,P]}\mathrm{e}^{-\langle \xi^\star, P\xi\rangle_Q} \tag{35}$$

is valid, where the sign $\epsilon_{[v,P]} = \mathrm{Pf}_{[v,J]}(\mathbb{1}-2P) = \pm 1$ depends on the orientation of the form v and the reflection $\mathbb{1}-2P$.

   This theorem states that in analogy to the bosonic expectation values of Weyl operators, the expectation values for the displacement operators for G-extended quasifree states are also of Gaussian type. In particular, the relation (35) is derived by calculating a Gaussian Grassmann integral with respect to the covariance $A = \mathbb{1}-2P$. Note that this notion of covariance is related to the Gaussian character of the Grassmann integral and should not be confused with the covariance operator that determines a quasifree fermion state. Recall that for a J-antisymmetric operator A on QH and a v normalized self-adjoint $2n$-form the corresponding Gaussian integral can be calculated according to

$$\int_Q v(\xi)\mathrm{e}^{(1/2)\langle \xi^\star, A\xi\rangle_Q + \langle \eta^\star, \xi\rangle_Q} = \mathrm{Pf}_{[v,J]}(A)\mathrm{e}^{(1/2)\langle \eta^\star, A^{-1}\eta\rangle_Q}. \tag{36}$$

Inserting $A = \mathbb{1}-2P = P - JPJ$ into the above identity (36) yields the identity (35).

G. Applications: Calculating the fidelity of quasifree states

   In this section, we give an explicit formula to calculate the fidelity between a pure quasifree state and another arbitrary quasifree state. Let us recall the fidelity between two states ω_1 and ω_2 on a general finite dimensional C*-algebra $\mathfrak{A}$. Let $L_2(\mathfrak{A})$ be the Hilbert space of Hilbert–Schmidt operators with respect to a faithful trace tr on $\mathfrak{A}$. A Hilbert–Schmidt operator $V \in L_2(\mathfrak{A})$ implements a state ω if $\omega(A) = \langle V, AV\rangle = \mathrm{tr}(V^*AV) = \mathrm{tr}(VV^*A)$ holds for all $A \in \mathfrak{A}$. In this case we write $v \in S(\omega)$. Clearly, for $V \in S(\omega)$ the operator VV^* is the density operator that corresponds to the state ω.

   As long as we consider finite dimensions, all states can be implemented that way. For two states ω, φ, the fidelity is given by

$$F(\omega, \varphi) = \sup_{V \in S(\omega), W \in S(\varphi)} |\langle V, W\rangle|. \tag{37}$$

As it has been shown by Bures[3] (compare also Ref. 16), the fidelity is related to the norm distance of states, independent of the dimension of the underlying algebra, by the inequality

$$\tfrac{1}{4}\|\omega - \varphi\|^2 \le 2(1 - F(\omega, \varphi)) \le \|\omega - \varphi\|. \tag{38}$$

The fidelity can simply be calculated if we choose one of the states ω to be pure. In this case, for each implementing Hilbert–Schmidt operator $V \in S(\omega)$ there exists a unitary $U \in \mathfrak{A}$, such that $V = EU$, where $E \in \mathfrak{A}$ is the unique rank-one projection (density operator) that corresponds to ω, i.e., $\omega(A) = \mathrm{tr}(EA)$. Thus we observe for each $W \in S(\varphi)$ that $\langle EU, W \rangle^2 = \mathrm{tr}(W^*EU)\mathrm{tr}(U^*EW) = \mathrm{tr}(UW^*E)\mathrm{tr}(EWU^*) = \mathrm{tr}(W^*EW) = \varphi(E)$ and we get for the fidelity

$$F(\omega, \varphi)^2 = \varphi(E). \tag{39}$$

As we have seen above, the GAR framework allows to calculate expectation values of operators in terms of Grassmann integrals. In the case of quasifree states, we see from Theorem 11 that we are faced here with calculating Gaussian Grassmann integrals only. This leads to the following theorem.

Theorem 12: *Let S be two covariance operator and let P be a basis projection on QH and let ω_S, ω_P be the corresponding quasifree states. Then the relation*

$$F(\omega_P, \omega_S)^2 = \left| \det(\mathbb{1} - P - S) \right|^{1/2} \tag{40}$$

is valid.

Proof: Since ω_P is a pure state, the square fidelity $F(\omega_P, \omega_S)^2 = \omega_S(E_P)$ is given by the expectation value of the support projection E_P in the state ω_S. To calculate the fidelity, we take advantage of Theorem 9 and Theorem 11 which can be used to express the expectation value of the support projection E_P in the state ω_S as a Gaussian Grassmann integral,

$$\omega_S(E_P)\mathbb{1} = \epsilon_{[v,P]} \int_Q v(\xi) e^{(1/2)\langle \xi^\star, (P-S)\xi \rangle_Q} e^{-\langle \xi^\star, P\xi \rangle_Q} = \epsilon_{[v,P]} \int_Q v(\xi) e^{(1/2)\langle \xi^\star, (\mathbb{1}-P-S)\xi \rangle_Q}, \tag{41}$$

where v is a normalized self-adjoint form of highest degree. We evaluate the integral with help of (36) which leads to

$$\omega_S(E_P) = \epsilon_{[v,P]} \, \mathrm{Pf}_{[v,J]}(\mathbb{1} - P - S). \tag{42}$$

Since the left hand side is positive (expectation value of a positive operator) and by the identity $\mathrm{Pf}_{[v,J]}(\mathbb{1}-P-S)^2 = \det(\mathbb{1}-P-S)$, we obtain the desired result. ∎

H. Finite versus infinite dimensions

To what extend can our formalism be used for infinite dimensional fermion systems? Recall that the dimension of the fermion system is given here by the dimension of the projection Q. The GAR algebra $\mathcal{G}(H, Q; J)$ is well defined for all separable Hilbert spaces and for all projections Q commuting with J. The anticommutative phase space for fermions as well as the Weyl–Grassmann operators can also be constructed in this case. The main difficulty to extend our kind of "harmonic analysis" to infinite dimensional systems is due to the problem of defining an appropriate infinite dimensional Grassmann integral. Of course, one can try to approximate (in some appropriate sense) an infinite Grassmann integral by a sequence of finite dimensional integrations. For instance, one can choose an increasing sequence of projections (E_n), $\lim E_n = \mathbb{1}$, that commute with Q and J and suppose further that $\dim(E_n Q) = 2n$. Now each of the anticommutative phase spaces $\mathcal{R}(E_n H, E_n Q; E_n J)$ can be identified with a subspace of $\mathcal{R}(H, Q; J)$. For infinite rank projections Q we propose to call a function f on $\mathcal{R}(H, Q; J)$ with values in a right module $\mathcal{E}$ to be G-holomorphic if for any finite rank projection E that commutes with Q and J the function $[f]_{EQ} : \mathcal{R}(EH, EQ; EJ) \ni \xi \rightarrow f(\xi)$ is G-holomorphic. A proposal for an infinite Grassmann integral is now to choose an appropriate sequence of forms $v = (v_n)$, where v_n is of highest degree with respect to $E_n Q$, and define

$$\int_Q \upsilon(\xi)f(\xi) := \lim_n \int_{E_n Q} \upsilon_n(\xi)f(\xi) \tag{43}$$

for an appropriate limit. The problem is here to find a reasonable form of convergence.

Concerning applications, one can directly ask whether a statement such as Theorem 12 can expected to be true for infinite dimensional fermion systems. Let P be a basis projection and S be a covariance operator on an infinite dimensional Hilbert space $K=QH$. Then the fidelity is only nonvanishing if the Gelfand-Naimark-Segal (GNS) representations of ω_S and ω_P are mutually quasiequivalent. To explain this, let π denote the GNS representation of ω_P, let $\mathcal{H}$ denote the GNS Hilbert space, and let Ω be the GNS (Fock vacuum) vector in $\mathcal{H}$, i.e., $\omega_P(A)=\langle\Omega,\pi(A)\Omega\rangle$ holds for all operators A in the fermion algebra. The quasiequivalence implies that there exists a density operator ρ on $\mathcal{H}$, such that $\omega_S(A)=\mathrm{tr}(\rho\pi(A))$ holds. A necessary and sufficient criterion for quasiequivalence has been shown by Araki (Ref. 2, Theorem 1) which states that the operator $\sqrt{S}-P$ has to be a Hilbert–Schmidt operator. A first natural question that arises here is whether the determinant $\det(\mathbb{1}-P-S)$ exists under these circumstances. Indeed the determinant of an operator $\mathbb{1}+A$ exists in infinite dimensions, provided A is a trace class operator (see, for instance, Ref. 18 Chap. XIII). Unfortunately, the operator $P+S$ is, in general, *not* trace class and the formula for the fidelity in Theorem 12 cannot hold in infinite dimensions. One possible way out of this dilemma is to "renormalize" the determinant with respect to the basis projection P. For instance, we multiply the operator $\mathbb{1}-P-S$ by the unitary $\mathbb{1}-2P$ from the right which yields $\mathbb{1}-P-S+2SP$. In finite dimensions the modulus of the determinant $|\det(\mathbb{1}-P-S)|=|\det(\mathbb{1}-P-S+2SP)|$ does not change and the determinant $\det(\mathbb{1}-P-S+2SP)$ is normalized in the sense that for $S=P$ we get $\det(\mathbb{1}-P-S+2SP)=\det(\mathbb{1})=1$. Now, if $\sqrt{S}-P$ is Hilbert–Schmidt, then it follows that $P+S-\{P,S\}$ is trace class. Thus for the simplified case that $[P,S]=0$ the "renormalized" determinant $\det(\mathbb{1}-P-S+2SP)$ exists in infinite dimensions. This shows that there might be an analogous determinant formula for the fidelity between a pure quasifree and an arbitrary quasifree state for the infinite dimensional case.

III. ON THE STRUCTURE OF THE GAR ALGEBRA

In this section, we discuss mathematical issues on the GAR algebra which are needed to prepare and derive the results which we have discussed in Sec. II F. Hereby we mainly focus on the algebraic and functional analytic properties. All the results that we derive here are also valid for the infinite dimensional (but separable) case.

A. Existence of the adjoint

As we have introduced in Sec. II B, the GAR algebra for a Hilbert space H, a complex conjugation J, and a projection Q that commutes with J is defined as the norm-closed subalgebra of the enveloping CAR algebra $\mathcal{E}(H,Q;J)$, that is, generated by the operators $G(f):=B(f\oplus 0)$. Therefore, by construction, the GAR algebra is a Banach algebra. As we have promised in Sec. II B, we show here that the GAR algebra admits a continuous adjoint that coincides with the C*-adjoint on the fermionic part.

Proposition 13:

(1) *There exists an adjoint* $\star:\mathcal{G}(H,Q;J)\to\mathcal{G}(H,Q;J)$ *, such that the GAR algebra becomes a Banach* $\star$ *-algebra.*
(2) *The adjoint* $\star$ *coincides with the* C* *-adjoint on the fermionic part:* $A^\star=A^*$ *for* $A\in\mathcal{F}(QH,QJ)$ *.*
(3) *The adjoint* $\star$ *is uniquely determined by the relation* $G(f)^\star=G(Jf)$ *for* $f\in H$ *.*

Proof: We consider the dense subalgebra $\mathring{\mathcal{G}}(H,Q;J)$ in $\mathcal{G}(H,Q;J)$ that consists of finite sums of finite products of operators $G(f)$ with $f\in H$. On this dense subalgebra, we define an anti-linear involution according to $G(f)^\star=G(Jf)$ for $f\in H$. To prove the proposition, we just have to show that this involution is continuous. For this purpose, we consider a finite rank projec-

tion E that commutes with Q and J. We obtain a closed finite dimensional subalgebra $\mathcal{G}(EH,EQ;EJ)\subset\mathring{\mathcal{G}}(H,Q;J)$ which is closed under the adjoint $\star$. Therefore, $\star$ is bounded on $\mathcal{G}(EH,EQ;EJ)$ which implies $\|A^\star\|=\|A\|$ for all $A\in\mathcal{G}(EH,EQ;EJ)$. Note that any bounded involution on a Banach space is isometric. Since each operator A in the GAR algebra is a norm convergent limit of a sequence of operators (A_n), where $A_n\in\mathcal{G}(E_nH,E_nQ;E_nJ)$ and (E_n) is an increasing sequence of projections that commute with J and Q, we conclude that the adjoint $\star$ is isometric on a norm dense subalgebra. Hence it is norm continuous and can uniquely be extended to the full GAR algebra. Since we have $G(Qf)^*=B(J_Q(Qf\oplus 0))=G(JQf)=G(Qf)^\star$, we conclude that $\star$ coincides with the C*-adjoint on the fermionic part. ∎

B. Natural norms on the Grassmann algebra

As a Banach *-algebra, the natural norm on the Grassmann algebra is the operator norm that is induced by the enveloping CAR algebra. In this norm the product is continuous which is the defining property of a Banach algebra norm.

As we have discussed in Sec. II B, the antisymmetric tensor algebra $\mathring{\Lambda}(H,J)$ over the Hilbert space H with complex conjugation J can be identified with a norm dense *-subalgebra in $\Lambda(H,J)$. In particular, $\mathring{\Lambda}(H,J)$ can also be identified with a dense subspace of the antisymmetric Fock space $F_-(H)$. Thus, $\mathring{\Lambda}(H,J)$ possesses a scalar product $(\lambda,\mu)\mapsto\langle\lambda,\mu\rangle$. The norm $|||\lambda|||=\sqrt{\langle\lambda,\lambda\rangle}$, which is induced by the scalar product, will be called the *Fock space norm*. The first observation is that the Fock space norm, and hence the scalar product, is continuous with respect to the Banach algebra norm as stated by the following proposition.

Proposition 14: The norm $|||\cdot|||$ is continuous with respect to the underlying Banach algebra norm $\|\cdot\|$. In particular, the scalar product is continuous and can uniquely be extended to the full Grassmann algebra.

Proof: By the identity $\mathring{\Lambda}(H,J)=\mathring{\mathcal{G}}(H,0;J)$ the enveloping CAR algebra is given by $\mathcal{F}(H\oplus H,J_0)$, where J_0 is the complex conjugation $J_0(h\oplus f)=Jf\oplus Jh$. We introduce the basis projection E on $H\oplus H$ according to $E(f\oplus h):=f\oplus 0$. The corresponding Fock representation of the enveloping C*-algebra π on $F_-(H)$ is faithful. The Grassmann field operators are represented in terms of the creation operators by $\pi(\Lambda(f))=\pi(B(f\oplus 0))=c^*(f)$. Clearly, the vectors $c^*(f_1)\cdots c^*(f_n)\Omega$ span a dense subspace in $F_-(H)$, where Ω is the corresponding Fock vacuum vector. Therefore, we have for an operator $\lambda\in\mathring{\Lambda}(H,J)$

$$|||\lambda|||=\|\pi(\lambda)\Omega\|\leq\|\lambda\|. \tag{44}$$

Hence the norm $|||\cdot|||$ is continuous with respect to $\|\cdot\|$. ∎

It is worth to mention that both norms are different from each other. In particular, the norm $|||\cdot|||$ is *not* a Banach algebra norm. This can be verified by the following counterexample: Take mutually orthogonal vectors $e_1,\ldots,e_6$ in H and consider the operator $\lambda=e_1\wedge e_2+e_3\wedge e_4+e_5\wedge e_6$. Then we find for the Fock space norm $|||\lambda|||=\sqrt{3}$, whereas a straightforward computation for the Fock space norm of λ^2 yields $|||\lambda^2|||=2\sqrt{3}>3=|||\lambda|||^2$.

C. Nilpotent ideals, positive operators, and positive functionals

As a *-algebra, $\mathcal{G}(H,Q;J)$ possesses a natural convex cone of positive elements: The set of positive linear functionals consists of all linear functionals $\omega:\mathcal{G}(H,Q;J)\to\mathbb{C}$ with $\omega(A^\star A)\geq 0$. The positive cone $\mathcal{G}(H,Q;J)_+$ consists of all operators that have positive expectation values for all positive functionals.

In order to analyze the positivity of operators, we introduce the norm-closed two-sided ideal $\mathcal{I}(H,Q;J)$, that is, generated by the *self-adjoint nilpotent operators* that belong to the Grassmann part $\Lambda(Q^\perp H,Q^\perp J)$. Recall that an operator Z is nilpotent if there exists $n\in\mathbb{N}$ with $Z^n=0$. The operators that are given by finite sums

$$A = \sum_{i=0}^{k} A_i Z_i \qquad (45)$$

with self-adjoint nilpotent operators $Z_i \in \Lambda(Q^{\perp}H, Q^{\perp}J)$ and $A_i \in \mathcal{G}(H,Q;J)$ form a norm dense subset in $\mathcal{I}(H,Q;J)$. It s not difficult to see that $A, A^{\star}$ as well as $A + A^{\star}$ are nilpotent operators in the GAR algebra. Here one takes advantage of the fact that the graded commutator of any operator with an operator from the Grassmann part vanishes.

Proposition 15:

(1) *There exists a surjective *-homomorphism $\epsilon_Q : \mathcal{G}(H,Q;J) \to \mathcal{F}(QH,QJ)$, such that the identities $\epsilon_Q(Z) = 0$ and $\epsilon_Q(A) = A$ are valid for all $Z \in \mathcal{I}(H,Q;J)$ and for all $A \in \mathcal{F}(QH,QJ)$.*
(2) *To each positive functionals ω on the GAR algebra $\mathcal{G}(H,Q;J)$ there exists a unique positive functional ω' on the fermionic part $\mathcal{F}(QH,QJ)$, such that*

$$\omega = \omega' \circ \epsilon_Q. \qquad (46)$$

Proof:

(1) In order to prove the existence of ϵ_Q, we show that the quotient algebra $\mathcal{G}(H,Q;J)/\mathcal{I}(H,Q;J)$ is canonically isomorphic to $\mathcal{F}(QH,QJ)$. Let π_Q be the canonical *-homomorphism that projects $\mathcal{G}(H,Q;J)$ onto $\mathcal{G}(H,Q;J)/\mathcal{I}(H,Q;J)$. The intersection $\mathcal{F}(QH,QJ) \cap \mathcal{I}(H,Q;J) = \{0\}$ only contains the zero element. Namely, let Z be self-adjoint and nilpotent, then $Z \in \mathcal{F}(QH,QJ)$ implies $Z = 0$ since the only self-adjoint and nilpotent element inside a C^{*}-algebra is the zero operator. For each generator $G(f)$ of the GAR algebra, we have the decomposition $G(f) = G(Qf) + G(Q^{\perp}f)$ with $G(Q^{\perp}f) \in \mathcal{I}(H,Q;J)$. Thus we conclude $\pi_Q(G(f)) = G(Qf) + \mathcal{I}(H,Q;J)$ and $\iota_Q : G(Qf) + \mathcal{I}(H,Q;J) \mapsto B(Qf) \in \mathcal{F}(QH,QJ)$ is the desired isomorphism. Thus $\epsilon_Q := \iota_Q \circ \pi_Q$ is a *-homomorphism that annihilates the ideal $\mathcal{I}(H,Q;J)$ and acts as the identity on the fermionic part.

(2) Let ω be a positive functional on the GAR algebra and let Z be self-adjoint and nilpotent. Then we can choose $k \in \mathbb{N}$, such that $Z^n = 0$ with $n = 2^k$. By iterating the Cauchy–Schwarz inequality, we conclude $|\omega(Z)| \leq \omega(Z^n) = 0$. Thus ω annihilates all nilpotent self-adjoint elements. Since the ideal $\mathcal{I}(H,Q;J)$ possesses a dense subsubspace that is spanned by nilpotent self-adjoint operators and since ω is continuous, the ideal $\mathcal{I}(H,Q;J)$ is annihilated which implies $\omega = \omega' \circ \epsilon_Q$, where ω' is the restriction of ω to the fermionic part. Thus each positive functional on the GAR algebra is the pull back of a unique (note that the dual map of ϵ_Q is injective) positive functional on the fermionic part via the *-homomorphism ϵ_Q. By construction we have $\omega(A + Z) = \omega'(\epsilon_Q(A + Z)) = \omega'(A)$. ∎

IV. ON GRASSMANN INTEGRALS

Toward the development of a "harmonic analysis" on antisymmetric phase space, we review here the basic concepts of Grassmann calculus, including integration theory. In view of our applications to fermionic systems, we need to give here a version which at some points differ from the standard analysis that can be found within the literature. In what follows, we assume here that the projection Q is of even finite rank $= 2n$, and that the rank of $Q^{\perp}$ is infinite.

A. G-holomorphic functions

In order to perform integration with respect to Grassmann variables we have to consider the algebra of functions that can be integrated. These functions are appropriate polynomials of Grassmann variables $\xi \in \mathcal{R}(H,Q;J)_0$ with values in a right module over the ring $\Lambda(Q^{\perp}H, Q^{\perp}J)$. If the underlying ring structure is clear from the context we just briefly say "right module." In the following, the right modules $\mathcal{E}$ under consideration are assumed to be Banach spaces with a continuous right multiplication.

Roughly, G-holomorphic functions are polynomials in the Grassmann variables. The problem is that the polynomial representation is not unique which causes ambiguities in the definition of the Grassmann integral. In order to overcome this problem, we define $\mathcal{E}_Q$ to be the norm-closed subrightmodule that consists of all elements X, such that $X\lambda_1\cdots\lambda_{2n}=0$ holds for all families $\lambda_1,\ldots,\lambda_{2n}$ of $2n$ odd operators in the Grassmann part ($2n$ is the rank of the projection Q). As already mentioned, the definition of the Grassmann integral is most comfortable in the case where the submodule $\mathcal{E}_Q=\{0\}$ is trivial. In this context, the most important example for such a right module is the GAR algebra for which the complementary projection $Q^\perp$ has infinite rank. The Grassmann part is isomorphic to the DeWitt algebra that is build from an infinite number of anticommuting generators.[7]

Proposition 16: Let Q be a projection of even and finite rank $2n$ and suppose that the rank of $Q^\perp$ is infinite. Then the closed subspace $\mathcal{G}(H,Q;J)_Q=\{0\}$ is trivial.

Proof: Let E be a finite rank projection that commutes with Q and J. Then the GAR algebra $\mathcal{G}(EH,EQ;EJ)$ is a finite dimensional subalgebra of $\mathcal{G}(H,Q;J)$. Using the enveloping CAR algebra, the full GAR algebra is isomorphic to the twisted tensor product $\mathcal{G}(H,Q;J)$ $=\mathcal{G}(EH,EQ;EJ)\widetilde{\otimes}\mathcal{G}(E^\perp H,E^\perp J;E^\perp J)$, where $\widetilde{\otimes}$ denotes the twisted tensor product. Note that the Banach algebra norm is a cross norm with respect to the twisted tensor product. Since $Q^\perp$ is infinite dimensional and E is finite dimensional, we conclude that $E^\perp Q^\perp=Q^\perp E^\perp$ is infinite dimensional. For each $A\in\mathcal{G}(EH,EQ;EJ)$ with $\|A\|>0$ we can choose odd Grassmann operators $\lambda_1,\ldots,\lambda_{2n}\in\Lambda(E^\perp Q^\perp H,E^\perp Q^\perp J)_1$, such that $\|\lambda_1\cdots\lambda_{2n}\|=1$. This implies that $\|A\lambda_1\cdots\lambda_{2n}\|$ $=\|A\|\|\lambda_1\cdots\lambda_{2n}\|=\|A\|$. Here we have used the fact that $A\lambda_1\cdots\lambda_{2n}\cong A\widetilde{\otimes}\lambda_1\cdots\lambda_{2n}$. For each nonzero $A\in\mathcal{G}(H,Q;J)$ and for each ϵ with $\|A\|>\epsilon>0$ we can find a finite rank projection E that commutes with Q and J and an operator $A_\epsilon\in\mathcal{G}(EH,EQ;EJ)$ with $\|A_\epsilon\|=\|A\|$, such that $\|A-A_\epsilon\|\leq\epsilon$. Again we can find odd Grassmann operators $\lambda_1,\ldots,\lambda_{2n}\in\Lambda(E^\perp Q^\perp H,E^\perp Q^\perp J)_1$, such that $[\lambda_1\cdots\lambda_{2n}]=1$. Suppose now that $A\lambda_1\cdots\lambda_{2n}=0$ then we conclude $\|A\lambda_1\cdots\lambda_{2n}-A_\epsilon\lambda_1\cdots\lambda_{2n}\|$ $=\|A_\epsilon\lambda_1\cdots\lambda_{2n}\|=\|A\|\leq\epsilon$ which contradicts the assumption $\|A\|>\epsilon$. Therefore, $A\lambda_1\cdots\lambda_{2n}\neq0$ which implies that the subspace $\mathcal{G}(H,Q;J)_Q=\{0\}$ is trivial. ∎

The vector space $\mathcal{O}(H,Q,J|\mathcal{E})$ of *G-holomorphic functions* with values in a right module $\mathcal{E}$ consists of all functions from $\mathcal{R}(H,Q;J)_0$ into $\mathcal{E}$ which can be build from linear combinations of monomial functions,

$$\xi\mapsto X\xi_{u_1}\cdots\xi_{u_n}, \tag{47}$$

with $X\in\mathcal{E}$ and $u_1,\ldots,u_n\in QH$. Here the "u-component" of ξ is defined as $\xi_u:=\langle u\otimes\mathbb{1},\xi\rangle_Q$. The algebra (ring) of "Grassmann-valued" G-holomorphic functions $\mathcal{O}(H,Q,J)$ is the algebra that is generated by the functions $\xi\mapsto\xi_u$ with $u\in QH$. The vector space, as defined above, is canonically equipped with a right module structure. For a G-holomorphic function we define the corresponding action by $(F\cdot\lambda)(\xi):=F(\xi)\lambda$. The G-holomorphic functions $\mathcal{O}(H,Q,J|\mathcal{E})$ with values in the right module $\mathcal{E}$ are also equipped with a $\mathcal{O}(H,Q,J)$ right module structure. Indeed, the space $\mathcal{O}(H,Q,J|\mathcal{E})$ is the right module over $\mathcal{O}(H,Q,J)$, generated by $\mathcal{E}$. Each G-holomorphic function admits a polynomial representation induced by a real orthonormal basis $(e^i)_{i\in N}$ of QH, that is, indexed by the ordered set $N=\{1,\ldots,2n\}$. Any vector $\xi\in\mathcal{R}(H,Q;J)_0$ can be expanded in this basis as $\xi=\Sigma_i\Lambda^i\xi_i$ with $\xi_i\in\Lambda(Q^\perp H,Q^\perp J)_1$. The corresponding polynomial expansion of a G-holomorphic function F is given by

$$F(\xi)=\sum F^I\xi_I, \tag{48}$$

with coefficients F^I in the right module $\mathcal{E}$. The monomial ξ_I which is associated with an ordered subset $I=\{i_1<i_2<\cdots<i_k\}\subset N$ is given by $\xi_I:=\xi_{i_1}\cdots\xi_{i_k}$. In particular, since Q has finite rank $2n$, each G-homomorphic function can be expressed as a finite sum of monomials, i.e., there is no problem concerning convergence.

There is an interesting connection between G-holomorphic functions and right module homomorphisms. To make this point clear, we observe that the Grassmann algebra $\Lambda(H,J)$ possesses a

natural right module structure over $\Lambda(Q^\perp H, Q^\perp J)$ by right multiplication $a \mapsto a\lambda$ with $a \in \Lambda(H,J)$ and $\lambda \in \Lambda(Q^\perp H, Q^\perp J)$. We denote by $\mathrm{Hom}(H,Q,J|\mathcal{E})$ the Banach space of bounded right module homomorphisms R from $\Lambda(H,J)$ into $\mathcal{E}$, i.e., R is complex linear and fulfills the condition $R(a\lambda) = R(a)\lambda$ for $a \in \Lambda(H,J)$ and $\lambda \in \Lambda(Q^\perp H, Q^\perp J)$. A particular case is here the Banach space of the right module homomorphisms with values in $\Lambda(Q^\perp H, Q^\perp J)$ which will be denoted by $\mathrm{End}(H,Q,J)$. Clearly, the space of right module homomorphism is a right module itself according to the following definition: An operator $\lambda \in \Lambda(Q^\perp H, Q^\perp J)$ acts on a right module homomorphism R as $(R \cdot \lambda)(a) := R(\lambda a)$.

For a G-holomorphic function $F \in \mathcal{O}(H,Q,J|\mathcal{E})$ we consider the subset $\mathcal{R}_Q F \subset \mathrm{Hom}(H,Q,J|\mathcal{E})$ that consists of all right module homomorphisms R, such that $F(\xi) = R(e^\xi)$ holds. To prepare our the definition of the Grassmann integral, we consider an operator $v \in \Lambda(QH,QJ)$ is called *a form of highest degree with respect to* Q if $v\Lambda(h) = 0$ for all $h \in QH$. If it is clear from the context to which projection Q we are referring, we shortly say that v is a form of highest degree. In general, we say that an operator is *a k-form with respect to* Q if it is a linear combination of operators of the form $\Lambda(Qf_1)\cdots\Lambda(Qf_k) = Qf_1 \wedge \cdots \wedge Qf_k$. Let n be the rank of the projection Q, then the subspace of k-forms has dimension $\binom{n}{k}$ and the subspace of forms of highest degree (n-forms) is a one dimensional.

Proposition 17: Let $F \in \mathcal{O}(H,Q,J|\mathcal{E})$ be a G-holomorphic function and let $R_1, R_2 \in \mathcal{R}_Q F \subset \mathrm{Hom}(H,Q,J|\mathcal{E})$ be two right module homomorphisms. Moreover, we assume that the submodule $\mathcal{E}_Q = \{0\}$ is trivial. Then the identity

$$R_1(v) = R_2(v) \tag{49}$$

holds for all forms v of highest degree.

Proof: Let $(e^i)_{i \in N}$ be a real basis of QH, indexed be the ordered set $N = \{1, 2, \ldots, n\}$. For each ordered subset $I \subset N$ we introduce the operator $\Lambda^I = \Lambda(e^{i_k})\cdots\Lambda(e^{i_1})$ with $I = \{i_1 < i_2 < \cdots < i_k\}$, where we put $\Lambda^\varnothing = 1$. Since $(\Lambda^I)_{I \subset N}$ is a basis of $\Lambda(QH,QJ)$, the operator e^ξ can be expanded as $\Sigma \Lambda^I \xi_I$ with $\xi_I = \xi_{i_1}\cdots\xi_{i_k}$. It follows from $F(\xi) = R_1(e^\xi) = R_2(e^\xi)$ that we obtain for the right module homomorphism $D = R_1 - R_2$

$$\sum_{I \subset N} D(\Lambda^I)\xi_I = 0 \tag{50}$$

for all $\xi \in \mathcal{R}(H,Q;J)$. From this we conclude that for each $I \subset N$ the identity

$$D(\Lambda^I)\xi_I = 0 \tag{51}$$

holds for all $\xi \in \mathcal{R}(H,Q;J)$. Since v is an operator of highest degree with respect to Q we have $v = v_N \Lambda^N$ which implies

$$D(v)\xi_N = 0 \tag{52}$$

for all ξ. But then $D(v)$ is contained in the submodule in $\mathcal{E}_Q = \{0\}$ which implies (49).   ■

B. Definition of the Grassmann integral and some basic properties

In the following discussion, we only consider right modules $\mathcal{E}$ for which the submodule $\mathcal{E}_Q = \{0\}$ is trivial. Let $v \neq 0$ be a nonzero form of highest degree. The Grassmann integral of F with respect to v is defined according to

$$\int_Q v(\xi)F(\xi) := R(v), \tag{53}$$

with an right module homomorphism $R \in \mathcal{R}_Q F$. Note that by Proposition 17, this definition only depends on the G-holomorphic function itself. The notation for the integral, as we use it, suggests to interpret the form of highest degree v as a volume form that is integrated over a noncommu-

tative space of Grassmann variables. The projection Q is interpreted as the realm of integration whose dimension is precisely the rank of Q. The symbolic expression $v(\xi)F(\xi)$ is then a volume form with values in the right module $\mathcal{E}$, evaluated at ξ.

Let $(e^i)_{i \in N}$ be a real orthonormal basis of QH. Each G-holomorphic function F with values in $\mathcal{E}$ can be expanded with respect to this basis as $F(\xi) = \Sigma_{I \subset N} F^I \xi_I$. Moreover, a form of highest degree $v \neq 0$ can be expressed in terms of this basis by $v = v_N \Lambda^N$,

This yields for the integral

$$\int_Q v(\xi)F(\xi) = \sum_{I \subset N} F^I v_I = v_N F^N. \tag{54}$$

This shows that our definition of the Grassmann integral is equivalent to the standard definition that can be found in the literature, see, for instance, Refs. 21 and 22.

The following proposition lists some basic and well known properties of the Grassmann integral. We also give here the proof, since our formalism (although equivalent) is a bit different from the one that can be found in the literature.

Proposition 18: Let v be an nonzero form of highest degree with respect to a projection Q of rank $2n$. Then the Grassmann integral has the following properties.

(1)   The Grassmann integral is translationally invariant: For each G-holomorphic function F with values in $\mathcal{E}$ the identity

$$\int_Q v(\xi)F(\xi) = \int_Q v(\xi)F(\xi + \eta) \tag{55}$$

holds for all $\eta \in \mathcal{R}(H,Q;J)_0$.

(2)   For a G-holomorphic function F with values in a right module $\mathcal{E}$ the identity

$$\int_Q vF \cdot \lambda = \left[\int_Q vF\right]\lambda \tag{56}$$

holds for all $\lambda \in \Lambda(Q^\perp H, Q^\perp J)$.

(3)   Let F be a G-holomorphic function with values in a right module $\mathcal{E}$ and let $T:\mathcal{E} \to \mathcal{E}'$ a right module homomorphism. For each G-holomorphic function F with values in $\mathcal{E}$ the function $TF:\xi \mapsto T(F(\xi))$ is G-holomorphic with values in $\mathcal{E}'$ and the identity

$$\int_Q vTF = T\left(\int_Q vF\right) \tag{57}$$

holds.

Proof: Recall that we have introduced the Grassmann integral with help of the space of right module homomorphisms $\mathcal{R}_Q F$.

(1)   For a right module homomorphism $R \in \mathcal{R}_Q F$, we obtain a right module homomorphism $\tau_\eta R \in \mathcal{R}_Q(\tau_\eta F)$ by putting $(\tau_\eta R)(a) := R(\exp(\eta)a)$. For an operator v of highest degree the integral of the translated function can be calculated by $\int_Q v\tau_\eta F = \tau_\eta R(v) = R(\exp(\eta)v)$. Since $v\Lambda(h) = 0$ for all $h \in QH$, it follows that $v\eta = 0$ for all $\eta \in \mathcal{R}(H,Q;J)$. This implies $\exp(\eta)v = v$ which yields the desired relation $\int_Q v\tau_\eta F = R(v) = \int_Q vF$.

(2)   Let F be a G-holomorphic function with values in $\mathcal{E}$. Then for $\lambda \in \Lambda(Q^\perp H, Q^\perp J)$ a right module homomorphism in $\mathcal{R}_Q(F \cdot \lambda)$ is simply given by $(R \cdot \lambda)(a) = R(\lambda a)$ with $R \in \mathcal{R}_Q F$. Thus we obtain for an operator v of highest degree $\int_Q vF \cdot \lambda = R(\lambda v) = R(v\lambda) = R(v)\lambda = [\int_Q vF]\lambda$.

(3)   Let $T:\mathcal{E} \to \mathcal{E}'$ be a right module homomorphism, then it is obvious that for a G-holomorphic function F with values in $\mathcal{E}$, the function $TF:\xi \mapsto T(F(\xi))$ is G-holomorphic with values in $\mathcal{E}'$. Since the identity $(T \circ R)(\exp(\xi)) = TF(\xi)$ is valid for all $R \in \mathcal{R}_Q F$, we conclude that

$T \circ R \in \mathcal{R}_Q TF$ which implies for an operator v of highest degree: $\int_Q v TF = (T \circ R)(v) = T(R(v)) = T(\int_Q v F)$. ∎

To treat multiple Grassmann integration, we have to say what is a G-holomorphic function in several variables: A function on $\mathcal{R}(H, Q; J)^n$ with values in $\mathcal{E}$ is called G-holomorphic if for each j the function

$$\xi \mapsto F(\xi_1, \ldots, \xi_{j-1}, \xi, \xi_{j+1}, \ldots, \xi_n) \tag{58}$$

is G-holomorphic. For our purpose, it is sufficient to consider the case $n = 2$. For a G-holomorphic function on $\mathcal{R}(H, Q; J)^2$ with values in a right module $\mathcal{E}$, we obtain two ordinary G-holomorphic functions F_η and F^ξ according to

$$F^\xi(\eta) := F_\eta(\xi) := F(\eta, \xi). \tag{59}$$

The following proposition can be used for exchanging the order of multiple Grassmann integrations.

Proposition 19: Let F be a G-holomorphic function on $\mathcal{R}(H, Q; J)^2$ with values in $\mathcal{E}$ and let v, w be forms of highest degree with respect to the even rank projection Q. Then the functions $F^v: \xi \mapsto \int_Q v F^\xi$ and $F_w: \eta \mapsto \int w F_\eta$ are G-holomorphic and the order of integration can be exchanged,

$$\int_Q v F_w = \int_Q w F^v. \tag{60}$$

Proof: Since F is G-holomorphic in both variables, we obtain from the polynomial expansion that there exists linear map $R: \Lambda(H, J)^2 \to \mathcal{E}$, such that $R(a\lambda, b) = R(a, \theta(b))\lambda$ for $\lambda \in \Lambda(Q^\perp H, Q^\perp J)_1$ and $R(a, b\lambda) = R(a, b)\lambda$ holds and that fulfills the identity $F(\eta, \xi) = R(e^\eta, e^\xi)$.

For fixed η, the map $R_\eta: b \mapsto R(e^\eta, b)$ is a right module homomorphism in $\mathcal{R}_Q F_\eta$. Thus, we obtain for the partial integral $F_w(\eta) = \int_Q w F_\eta = R(e^\eta, w)$ which also shows that F_w is G-holomorphic. Since w is even, it follows that the map $R_w: a \mapsto R(a, w)$ is a right module homomorphism in $\mathcal{R}_Q F_w$ which implies $\int_Q v F_w = R(v, w)$. By a similar argument one shows that $\int w F^v = R(v, w)$ which implies the result. ∎

C. Some useful lemmas for calculating Grassmann integrals

In order to calculate particular Grassmann integrals, a further interesting fact to mention is that the algebra of G-holomorphic functions is related to the Grassmann algebra $\Lambda(H, J)$ itself. To explain this, we observe that the Grassmann algebra (as a Banach space) $\Lambda(H, J)$ is isomorphic to the tensor product $F_-(QH) \otimes \Lambda(Q^\perp H, Q^\perp J)$. The canonical isomorphism is given by identifying

$$\Lambda(f_1) \cdots \Lambda(f_n)\lambda \cong f_1 \wedge \cdots \wedge f_n \otimes \lambda \tag{61}$$

for $f_1, \ldots, f_n \in QH$ and $\lambda \in \Lambda(Q^\perp H, Q^\perp J)$. This can be used to introduce a rigging map on $\Lambda(H, J)$ with values $\Lambda(Q^\perp H, Q^\perp J)$. By using the induced Hilbert space structure on the Grassmann algebra $\Lambda(QH, QJ) \cong F_-(QH)$ the rigging map is determined by

$$\langle \Lambda(f_1) \cdots \Lambda(f_n)\lambda, \Lambda(h_1) \cdots \Lambda(h_n)\mu \rangle_Q = \langle f_1 \wedge \cdots \wedge f_n, h_1 \wedge \cdots \wedge h_n \rangle \lambda^\star \mu. \tag{62}$$

If we expand the operators in $\Lambda(H, J)$ in a real orthonormal basis $(e^i)_{i \in N}$ of QH, then the rigging map can be simply calculated as

$$\langle a, b \rangle_Q = \sum_{I \subset N} a_I^\star b_I, \tag{63}$$

where $a = \Sigma_{I \subset N} \Lambda^I a_I$ and $b = \Sigma_{I \subset N} \Lambda^I b_I$. We associate with each operator $a \in \Lambda(H, J)$ a G-holomorphic function according to

$$E_Q a(\xi) := \langle a^\star, \exp(\xi) \rangle_Q, \tag{64}$$

and we show the following useful lemma.

Lemma 20: The map $E_Q : \Lambda(H,J) \to \mathcal{O}(H,Q,J)$ is an algebra homomorphism. Moreover, for each $a \in \Lambda(H,J)$, the right module homomorphism $R_Q a : b \mapsto \langle a^\star, b \rangle_Q$ is contained in $\mathcal{R}_Q E_Q a$.

Proof: Let $(e^i)_{i \in N}$ be a real orthonormal basis of QH. Then we expand the operator $a \in \Lambda(H,J)$ according to $a = \Sigma_{I \subset N} a^I \Lambda_I$, where the product $\Lambda_I = \Lambda(e^{i_1}) \cdots \Lambda(e^{i_k})$ is ordered by increasing indices $I = \{i_1 < i_2 < \cdots < i_k$ (indicated by a subscript I). Moreover, the exponential $\exp(\xi)$ has an expansion $\exp(\xi) = \Sigma_{I \subset N} \Lambda^I \xi_I$, where the product $\Lambda^I = \Lambda(e^{i_k}) \cdots \Lambda(e^{i_1})$ is ordered by decreasing indices $I = \{i_1 < i_2 < \cdots < i_k$ (indicated by a superscript I). Since $a^\star = \Sigma_{I \subset N} \Lambda^I (a^I)^\star$ we find

$$E_Q a(\xi) := \sum_{I \subset N} a^I \xi_I. \tag{65}$$

For the product $E_Q a(\xi) E_Q b(\xi)$ we obtain the expansion

$$E_Q a(\xi) E_Q b(\xi) = \sum_{I,J \subset N} a^I \xi_I b^J \xi_J = \sum_{I \subset J \subset N} \epsilon_{IJ} a^I \theta^{|J \setminus I|} (b^{J \setminus I}) \xi_J, \tag{66}$$

where ϵ_{IJ} is the sign of the permutation $(I, J \setminus I) \to J$ which emerges from the identity $\xi_I \xi_{J \setminus I} = \epsilon_{IJ} \xi_J$. On the other hand, the product ab has the following expansion with respect to the chosen basis:

$$ab = \sum_{I,J \subset N} a^I \Lambda_I b^J \Lambda_J = \sum_{I \subset J \subset N} \epsilon_{IJ} a^I \theta^{|J \setminus I|} (b^{J \setminus I}) \Lambda_J, \tag{67}$$

which implies the homomorphism property. Finally, we directly observe that $E_Q a(\xi) = R_Q a(e^\xi)$ which concludes the proof. ∎

Lemma 21: Let F be a G-holomorphic function with values in a right module $\mathcal{E}$ and let v_1, v_2 be two operators of highest degree with respect to Q. Then the identity

$$\int_Q v_1(\xi) \int_Q v_2(\eta) F(\xi) e^{\langle \eta^\star, \xi - \zeta \rangle_Q} = \langle v_1^\star, v_2 \rangle F(\zeta) \tag{68}$$

is valid.

Proof: We take advantage of the fact that the bilinear form $(a,b) \mapsto \langle a^\star, b \rangle_Q$ fulfills the identity $\langle a^\star, b \rangle_Q = \langle b^\star, \theta_Q(a) \rangle_Q$ on the even subalgebra. Here θ_Q is the automorphism $\theta_Q(\Lambda(f)) = \Lambda((1-2Q)f)$. From this we calculate the following Grassmann integral: Let v a form of highest degree with respect to a projection Q of even rank n. Then we calculate

$$\int_Q v(\xi) e^{\langle \eta^\star, \xi \rangle_Q} = \langle e^{\eta^\star}, v \rangle_Q = \frac{1}{n!} \langle v^\star, \eta^n \rangle_Q. \tag{69}$$

Furthermore, the value of the rigging map $\langle a, b \rangle_Q = \langle a, b \rangle \mathbb{1}$ is a multiple of the identity for all operators a, b that belong to the subalgebra $\Lambda(QH, QJ)$. As a consequence we can calculate the double integral,

$$\int_Q v_1(\eta) \int_Q v_2(\xi) e^{\langle \eta^\star, \xi \rangle_Q} = \int_Q v_1(\eta) \langle v_2^\star, e^\eta \rangle_Q = \langle v_1^\star, v_2 \rangle \mathbb{1}. \tag{70}$$

Finally, we introduce the function δ_v that is given by $\delta_v(\eta) := (n!)^{-1} \langle v^\star, \eta^n \rangle_Q$ with $n = \dim(Q)$. As we will see, the G-holomorphic function δ_v plays the role of a δ-function for Grassmann integrals for the "volume form" v. To verify this, we observe that $\eta \delta_v(\eta) = 0$ holds and we calculate for a G-holomorphic function F the multiple integral,

$$\int_Q v_1(\xi)\left[\int_Q v_2(\eta)F(\xi)e^{\langle\eta^\star,\xi-\zeta\rangle_Q}\right] = \int_Q v_1(\xi)F(\xi)\delta_{v_2}(\xi-\zeta). \tag{71}$$

By taking advantage of the translation invariance of the Grassmann integral, it follows that

$$\int_Q v_1(\xi)\left[\int_Q v_2(\eta)F(\xi)e^{\langle\eta^\star,\xi-\zeta\rangle_Q}\right] = \int_Q v_1(\xi)\tau_\zeta F(\xi)\delta_{v_2}(\xi) \tag{72}$$

holds. Since $\xi\delta_{v_2}(\xi)=0$, we find for each $R\in\mathcal{R}_Q F$

$$F(\xi)\delta_{v_2}(\xi) = R[e^\xi\delta_{v_2}(\xi)] = R(\mathbb{1}) = F(0). \tag{73}$$

Putting all these together implies the δ-function formula,

$$\int_Q v_1(\xi)\left[\int_Q v_2(\eta)F(\xi)e^{\langle\eta^\star,\xi-\zeta\rangle_Q}\right] = \tau_\zeta F(0)\int_Q v_1(\xi)\delta_{v_2}(\xi) = \langle v_1^\star, v_2\rangle F(\zeta). \tag{74}$$

∎

D. Calculating Gaussian integrals

Let A be a bounded operator on QH, then there exists an operator $a\in\Lambda(QH,QJ)$, such that

$$\langle\xi^\star, A\xi\rangle_Q = 2\langle a^\star, \exp(\xi)\rangle_Q, \tag{75}$$

where a only depends on the antisymmetric part $(A-JA^*J)/2$ of A. In particular, a has degree of 2 with respect to Q. To verify this, we recall that the map $f\otimes h\mapsto\langle Jf,Ah\rangle$ is a bilinear form on QH. Since Q has finite rank, it follows that there exists a unique vector $\psi_A\in QH^{\otimes 2}$, such that $\langle(J\otimes J)\psi_A, f\otimes h\rangle=\langle Jf,Ah\rangle$ holds. Moreover, we have $\langle Jh,Af\rangle=\langle Jf,JA^*Jh\rangle=\langle(J\otimes J)\mathbb{F}\psi_A, f\otimes h\rangle$, where $\mathbb{F}$ is the flip operator that swaps the tensor product. The antisymmetric vector $a=(\mathbb{F}\psi_A-\psi_A)/2$ can be identified with an operator in $\Lambda(H,J)$, and we find

$$\langle a^\star, f\wedge h\rangle = \tfrac{1}{2}\langle Jf,(JA^*J-A)h\rangle. \tag{76}$$

According to the definition of the rigging map, we obtain for $\lambda,\mu\in\Lambda(Q^\perp H,Q^\perp J)_1$ and for $f,h\in QH$

$$\langle a^\star, \Lambda(f)\lambda\Lambda(h)\mu\rangle_Q = \tfrac{1}{2}\langle Jf,(A-JA^*J)h\rangle\lambda\mu, \tag{77}$$

which implies the identity (75). In the following, we can therefore restrict the consideration to operators on QH that fulfill the condition $A=-JA^*J$.

Since the operator a has degree of 2 the operator $(n!)^{-1}a^n$ with $2n=\dim(Q)$ is of highest degree and therefore proportional to any other operator of highest degree. We choose a self-adjoint operator of highest degree $v=v^\star$ that is normalized $\langle v,v\rangle=1$. Then the identity

$$(n!)^{-1}a^n = (n!)^{-1}\langle v,a^n\rangle v \tag{78}$$

holds. If a is related to an operator A on QB according to $\langle Jf,Ah\rangle=\langle a^\star, f\wedge h\rangle$, then the scalar product $(n!)^{-1}\langle v,a^n\rangle$ is the Pfaffian of A with respect to v and J (see Sec. II F),

$$\mathrm{Pf}_{[v,J]}(A) = (n!)^{-1}\langle v,a^n\rangle. \tag{79}$$

This can now be applied calculate Gaussian Grassmann integrals easily as stated by the next lemma.

Lemma 22: Let A be an antisymmetric operator on QH, i.e., $A=-JA^*J$, and let v be a self-adjoint normalized form of highest degree with respect to Q. Then the Gaussian integral identity,

$$\int_Q v(\xi) e^{(1/2)\langle \xi^\star, A\xi \rangle_Q} = \mathrm{Pf}_{[v,J]}(A)\mathbb{1},\tag{80}$$

holds.

Proof: Let $a \in \Lambda(QH, QJ)$ be the operator of degree of 2 that fulfills the identity $\langle a^\star, f \wedge h \rangle = \langle Jf, Ah \rangle$. Then we calculate the Gaussian integral with the help of Lemma 20 according to

$$\int_Q v(\xi) e^{(1/2)\langle \xi^\star, A\xi \rangle_Q} = \int_Q v(\xi) e^{\langle a^\star, e^\xi \rangle_Q} = \int_Q v(\xi)\langle e^{a^\star}, e^\xi \rangle_Q = \langle e^{a^\star}, v \rangle_Q.\tag{81}$$

Since both operators e^a and $v = v^\star$ are even and contained in $\Lambda(QH, QJ)$, we obtain for the rigging map $\langle e^{a^\star}, v \rangle_Q = \langle e^{a^\star}, v \rangle\mathbb{1} = \langle v, e^a \rangle\mathbb{1}$. By expanding the exponential e^a only the contribution to the operator of highest degree contributes to the scalar product. Therefore, we obtain

$$\langle v, e^a \rangle = (n!)^{-1}\langle v, a^n \rangle = \mathrm{Pf}_v(a) = \mathrm{Pf}_{[v,J]}(A),\tag{82}$$

which proves the lemma. ∎

The identity (36) can now be shown by using the translation invariance of the Grassmann integral. If $A = -JA^*J$ is invertible, we find

$$\langle (\xi - A^{-1}\eta)^\star, A(\xi - A^{-1}\eta) \rangle_Q = \langle \xi^\star, A\xi \rangle_Q - \langle \eta^\star, A^{-1}\eta \rangle_Q + 2\langle \eta^\star, \xi \rangle_Q,\tag{83}$$

where we have used the fact that $(A^{-1}\eta)^\star = JA^{-1}J\eta^\star$ holds. As a result we obtain from the previous lemma of highest degree with respect to Q. Then the Gaussian integral identity,

$$\int_Q v(\xi) e^{(1/2)\langle \xi^\star, A\xi \rangle_Q + \langle \eta^\star, \xi \rangle_Q} = \mathrm{Pf}_{[v,J]}(A) e^{(1/2)\langle \eta^\star, A^{-1}\eta \rangle_Q}.\tag{84}$$

In the case where A does not fulfill the antisymmetry condition $A = -JA^*J$, the Gaussian integral formula is still valid by substituting A on the right hand side by the antisymmetrized operator $(A - JA^*J)/2$.

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APPENDIX A: PROOF OF THEOREM 8

Proof: In the first step, we show that each operator A in the GAR algebra can be represented by a Grassmann integral,

$$A = \int_Q v(\xi)\mathbf{w}(\xi)f(\xi),\tag{A1}$$

with a G-holomorphic function f depending on v and A. Let $(e^i)_{i \in N}$ be a real orthonormal basis of QH with $N = \{1, 2, \ldots, \dim(Q)\}$. Then the Grassmann–Weyl operator can be expanded by

$$\mathbf{w}(\xi) = \sum_{I \subset N} B^I \xi_I,\tag{A2}$$

where we have introduced the operators $B^I = G(e^{i_k}) \cdots G(e^{i_1})$ for each ordered subset $I = \{i_1 < i_2 < \cdots < i_k\}$. For each subset $K \subset N$ we introduce the G-holomorphic function b^K by $b^K(\xi) := \epsilon_{KN}\xi_{N \setminus J}$, where ϵ_{KN} is determined by the condition $\epsilon_{KN}\xi_K\xi_{X \setminus N} = \xi_N$. Then we conclude from the polynomial expansion of the reduced Grassmann–Weyl operator $\mathbf{w}(\xi)$ that the identity

$$\mathbf{w}(\xi)b^K(\xi) = \sum_{I \subset N} B^I \xi_I \xi_{N \setminus K} = \sum_{I \subset K} \epsilon_{IKN} \epsilon_{KN} B^I \xi_{N \setminus (K \setminus I)} \tag{A3}$$

holds with $\xi_I \xi_{N \setminus K} = \epsilon_{IKN} \xi_{N \setminus (K \setminus I)}$. Note that for $I \not\subset K$ we have $\xi_I \xi_{N \setminus K} = 0$. This implies for a form v of highest degree,

$$\int_Q v(\xi) \mathbf{w}(\xi) b^J(\xi) = \int_Q v(\xi) \sum_{I \subset K} \epsilon_{IKN} \epsilon_{KN} B^I \xi_{N \setminus (K \setminus I)} = v_N B^K, \tag{A4}$$

where we have used the identity $\epsilon_{KKN} = \epsilon_{KN}$. The operators B^K form a basis of the fermionic part of the reduced GAR algebra. Thus a general operator can by expanded as $A = \Sigma_{I \subset N} B^I A_I$ with A_I belonging to the Grassmann part. Thus the G-holomorphic function,

$$f(\xi) = \sum_{I \subset N} v_N^{-1} \epsilon_{IN} \xi_{N \setminus I} A_I, \tag{A5}$$

solves the identity (26).

We insert now this identity into the Fourier transform and obtain

$$(\mathcal{F}A)(\xi) = \mathbf{w}(-\xi) \int_Q v(\eta) \int_Q v(\zeta) \mathbf{w}(-\eta) \mathbf{w}(\zeta) f(\zeta) \mathbf{w}(\eta) \exp(\langle \eta^\star, \xi \rangle_Q). \tag{A6}$$

By taking advantage of the fact that the Weyl operators belong to the even part of the GAR algebra, we obtain from the Weyl relations

$$(\mathcal{F}A)(\xi) = \mathbf{w}(-\xi) \int_Q v(\eta) \int_Q v(\zeta) \mathbf{w}(\zeta) f(\zeta) \exp(\langle \eta^\star, \xi - \zeta \rangle_Q). \tag{A7}$$

Since Q is even, the order of integration can be exchanged. Therefore, we get from the δ-function formula (68)

$$(\mathcal{F}A)(\xi) = \mathbf{w}(-\xi) \int_Q v(\zeta) \mathbf{w}(\zeta) f(\zeta) \int_Q v(\eta) \exp(\langle \eta^\star, \xi - \zeta \rangle_Q) = \langle v^\star, v \rangle f(\xi). \tag{A8}$$

By choosing v to be normalized and self-adjoint, implies the result.   ∎

APPENDIX B: PROOF OF THEOREM 9

Proof: We first consider the convolution of two G-holomorphic functions f, f' which is given by

$$(f * f')(\xi) = \int_Q v(\eta) f(\eta) f'(\xi - \eta). \tag{B1}$$

Taking the Fourier transform yields

$$\mathcal{F}(f * f')(\xi) = \int_Q v(\zeta) \int_Q v(\eta) f(\eta) f'(\zeta - \eta) e^{\langle \xi^\star, \zeta \rangle_Q}. \tag{B2}$$

We make use of the translation invariance of the Grassmann integral (Proposition 18) and the fact that the order of integration can be exchanged (Proposition 19) which implies

$$\mathcal{F}(f * f')(\xi) = \int_Q \upsilon(\eta)f(\eta)e^{\langle\xi^\star,\eta\rangle_Q} \int_Q \upsilon(\zeta)f'(\zeta)e^{\langle\xi^\star,\zeta\rangle_Q} = \mathcal{F}f(\xi)\mathcal{F}f'(\xi). \tag{B3}$$

Let φ be a bounded right module homomorphism from the GAR algebra into its Grassmann part and let A be an operator of the reduced Grassmann algebra. The convolution of φ and A is just given by $(\varphi * A)(\xi) = \varphi(\alpha_\xi A)$. Therefore, the Fourier transform of this convolution is

$$\mathcal{F}(\varphi * A)(\xi) = \int_Q \upsilon(\eta)\varphi(\alpha_{-\eta}A)e^{\langle\xi^\star,\eta\rangle_Q} = \varphi\left(\int_Q \upsilon(\eta)\alpha_{-\eta}Ae^{\langle\xi^\star,\eta\rangle_Q}\right). \tag{B4}$$

By Proposition 18, the integration and the application by the right module homomorphism can be exchanged. Keeping the definition of the Fourier transform of a GAR operator in mind, we obtain

$$\mathcal{F}(\varphi * A)(\xi) = \varphi(\mathbf{w}(\xi)\mathcal{F}A(\xi)) = \varphi(\mathbf{w}(\xi))\mathcal{F}A(\xi) = \mathcal{F}\varphi(\xi)\mathcal{F}A(\xi). \tag{B5}$$

In the last step, we have used the property that $\mathcal{F}A(\xi)$ is contained in the reduced Grassmann part and that φ is right module homomorphism.

Finally, we consider the convolution of an operator A in the reduced GAR algebra with a G-holomorphic function f. The Fourier transform is given by

$$\mathcal{F}(A * f)(\xi) = \int_Q \upsilon(\eta)\alpha_{-\eta}\left(\int_Q \upsilon(\zeta)\alpha_\xi Af(\zeta)\right)e^{\langle\xi^\star,\eta\rangle_Q} = \int_Q \upsilon(\eta)\int_Q \upsilon(\zeta)\alpha_{\zeta-\eta}Af(\zeta)e^{\langle\eta^\star,\xi\rangle_Q}. \tag{B6}$$

By using the translation invariance as well as the exchange rule for the order of integration, the identity $\mathcal{F}(A * f) = \mathcal{F}A\mathcal{F}f$ follows. ∎

APPENDIX C: PROOF OF THEOREM 11

Proof: To start with, we first look at the left hand side of Eq. (34),

$$\mathcal{F}\boldsymbol{\omega}_S(\xi) = e^{-(1/2)\langle\xi^\star,S\xi\rangle_Q}, \tag{C1}$$

that we have to show. For a real basis $(e^i)_{i \in N}$ of QH, the Grassmann–Weyl operator can be expanded as a polynomial in the Grassmann variable ξ. The Fourier transform (characteristic function) of the extended quasifree state $\boldsymbol{\omega}_S$ can be calculated by

$$\mathcal{F}\boldsymbol{\omega}_S(\xi) = \sum_{K \subset N} \omega_S(B^K)\xi_K, \tag{C2}$$

where B^K and ξ_K are defined as within the proof of Theorem 8 above. We obtain a polynomial expansion with complex valued coefficients, given by the quasifree expectation values $\omega_S(B^K)$. These expectation values can be calculated by Wick's theorem according to

$$\omega_S(B^K) = \sum_{\Pi \in P_2(K)} \epsilon_{\Pi K} \prod_{I \in \Pi} \omega_S(B^I), \tag{C3}$$

where $P_2(K)$ is the set of all ordered partitions of K into two-elementary subsets and $\epsilon_{\Pi K}$ is the sign of the permutation $(I_1, \ldots, I_k) \to K$ with $\Pi = (I_1, \ldots, I_k)$. This yields for the full polynomial expansion,

$$\boldsymbol{\omega}_S(\mathbf{w}(\xi)) = \sum_{K \subset N} \sum_{\Pi \in P_2(K)} \epsilon_{\Pi K} \prod_{I \in \Pi} \omega_S(B^I)\xi_K. \tag{C4}$$

On the other hand, there exists an operator a_S in $\Lambda(QH, QJ)$ of degree of 2 which is determined by the condition $2\langle a_S^\star, f \wedge h\rangle = \langle Jf, Sh\rangle$, $f, h \in QH$, and which satisfies the identity

$$\langle e^{a_S^\star}, e^\xi \rangle_Q = e^{-(1/2)\langle \xi^\star, S\xi \rangle_Q}. \tag{C5}$$

If we expand the exponential of a_S with respect to a real basis of QH, then we get, according to the calculations we have done in the proof of Lemma 20,

$$e^{a_S} = \sum_{K \subset N} \sum_{\Pi \subset P_2(K)} \epsilon_{\Pi K} \prod_{I \in \Pi} a_S^I \Lambda_K. \tag{C6}$$

This implies by using the expansion of the (reduced) rigging map,

$$e^{-(1/2)\langle \xi^\star, S\xi \rangle_Q} = \sum_{K \subset N} \sum_{\Pi \subset P_2(K)} \epsilon_{\Pi K} \prod_{I \in \Pi} a_S^I \xi_K. \tag{C7}$$

By construction, the identity $a_S^{\{i<j\}} = \langle e^i, Se^j \rangle = \omega_S(G(e^i)G(e^j))$ holds for all two-elementary ordered subsets $I = \{i < j\}$ which implies (34).

Let P be a basis projection on QH and let v be a self-adjoint normalized form of highest degree in $\Lambda(QH, QJ)$. Moreover, E_P denotes the support projection of the pure quasifree state ω_P. We now have to show the identity

$$\mathcal{F}_v(\boldsymbol{\omega}_P * E_P)(\xi) = e^{-\langle \xi^\star, P\xi \rangle_Q}. \tag{C8}$$

According to the definition of the convolution and the Fourier transform, we have to calculate the Grassmann integral,

$$\mathcal{F}_v(\boldsymbol{\omega}_P * E_P)(\xi) = \int_Q v(\eta) \boldsymbol{\omega}_P(\mathbf{w}(-\eta) E_P \mathbf{w}(\eta)) e^{\langle \xi^\star, \eta \rangle_Q}. \tag{C9}$$

Since ω_P is pure, the supposed projection E_P has rank one and the identity $\omega_P(AE_PB) = \omega_P(A)\omega_P(B)$ holds for A, B in the fermionic part. This property is lifted to the G-extension, i.e., $\boldsymbol{\omega}_P(AE_PB) = \boldsymbol{\omega}_P(A)\boldsymbol{\omega}_P(B)$. Since the equivalence class mapping $[\cdot]_Q$ is a *-algebra homomorphism, we obtain for the lift $\boldsymbol{\omega}_P$ to the reduced GAR algebra that $\boldsymbol{\omega}_P(AE_PB) = \boldsymbol{\omega}_P(A)\boldsymbol{\omega}_P(B)$ holds for all operators A, B in the reduced GAR algebra. To verify this we choose operators of the form $A\lambda$ and $B\mu$, where A, B belong to the fermionic part and $\lambda \in \Lambda(Q^\perp H, Q^\perp J)_q$, μ belong to the Grassmann part of the GAR algebra. Then we calculate

$$\boldsymbol{\omega}_P(A\lambda E_P B\mu) = \boldsymbol{\omega}_P(AE_P \theta^q(B)\lambda\mu) = \omega_P(AE_P \theta^q(B))\lambda\mu = \omega_P(A)\omega_P(B)\lambda\mu = \boldsymbol{\omega}_P(A\lambda)\boldsymbol{\omega}_P(B\mu). \tag{C10}$$

By using the identity (34), which we just have proven above, we get

$$\boldsymbol{\omega}_P(\mathbf{w}(-\eta)E_P\mathbf{w}(\eta)) = \boldsymbol{\omega}_P(\mathbf{w}(\eta))^2 = e^{-\langle \eta^\star, P\eta \rangle_Q}. \tag{C11}$$

By inserting this into Eq. (C9), it remains to calculate the Gaussian integral with help of Lemma 22 and the discussion thereafter in Sec. IV D,

$$\mathcal{F}_v(\boldsymbol{\omega}_P * E_P)(\xi) = \int_Q v(\eta) e^{-\langle \eta^\star, P\eta \rangle_Q + \langle \xi^\star, \eta \rangle_Q} = \mathrm{Pf}_{[v, J]}(JPJ - P) e^{(1/2)\langle \xi^\star, (JPJ-P)^{-1}\xi \rangle_Q}. \tag{C12}$$

Since P is a basis projection, the operator $JPJ - P = \mathbb{1} - 2P$ is a reflection and we have $\mathrm{Pf}_{[v, J]}(\mathbb{1} - 2P) = \pm 1$. Moreover, we conclude $\langle \xi^\star, (JPJ-P)^{-1}\xi \rangle_Q = \langle \xi^\star, \mathbb{1}(\mathbb{1}-2P)\xi \rangle_Q = -2\langle \xi^\star, P\xi \rangle_Q$. This yields the desired result. ∎

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