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SPECIAL ISSUE PAPER

Spin network quantum circuits

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SUMMARY

In this manuscript, a recent approach to quantum computation promoted by the authors, based on the theory of recoupling of quantum angular momenta instead of the conventional notion of q-bit (that simply mimicks the Booleam structure of the classical approach to computation), is reviewed and analyzed in its basic elements. In particular, the reach of the new scheme in terms of algorithmic complexity is discussed, focusing the attention on the quantum algorithm for generalized Jones polynomials of knots, $\mathrm{J}(L; q; j_1, j_2, \dots, j_M)$, because the problem of approximating $\mathrm{J}(L; q; \{j_i\})$ has been recognized to be a universal problem, namely the hardest problem that a quantum computer can efficiently handle.

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1. INTRODUCTION

The traditional dynamical evolution process associated with generic quantum information manipulation is grounded in the notion of *qu-bit*, out of which gates and circuits are described, essentially mimicking the classical Boolean representation. Qu-bits are nothing but *spin*-$\frac{1}{2}$ variables in $\mathcal{H}^{(\frac{1}{2})}$, Hilbert space isomorphic to $\mathbb{C}^2$, whose elements can be expressed as $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$. In this standard scheme, the computational gates are realized in terms of 2×2 complex unitary matrices (Pauli matrices) that belong to the fundamental irreducible representations $\mathcal{D}^{(\frac{1}{2})}$ of $SU(2)$.

The conceptual scheme proposed in this paper can be thought of as the non-Boolean generalization of the standard *quantum circuit* model, in which unitary gates are of two types: j-gates expressed in terms of $3nj$ coefficients connecting inequivalent binary coupling schemes of $n+1$ angular momentum variables, in which the qu-bit space is replaced by $\mathbb{C}^{2J+1}$, and M-gates that are Wigner rotations in the eigenspace of the total angular momentum $\mathbf{J}$. The non-Boolean character is twofold: computational states are labeled both by a discrete, finite number $2J+1$ of variables and by a set of continuous variables (see App. A). This representation implies that computational gates are mathematically realized in terms of more complex operators, just the $3nj$ symbols, belonging to the (re)coupling theory of $SU(2)$ for generic, arbitrary angular momenta.

Such scheme – that we refer to as *Spin Network Quantum Automaton* (SNQA) – automatically incorporates all the essential features that make quantum information encoding so much more efficient than the classical one: it is fully discrete; it deals with inherently entangled states; it is naturally endowed with a tensor product structure; and it allows for generic encoding patterns. Elementary j-gates are particularly important, as they are represented by $6j$ symbols, which satisfy algebraic identities that make

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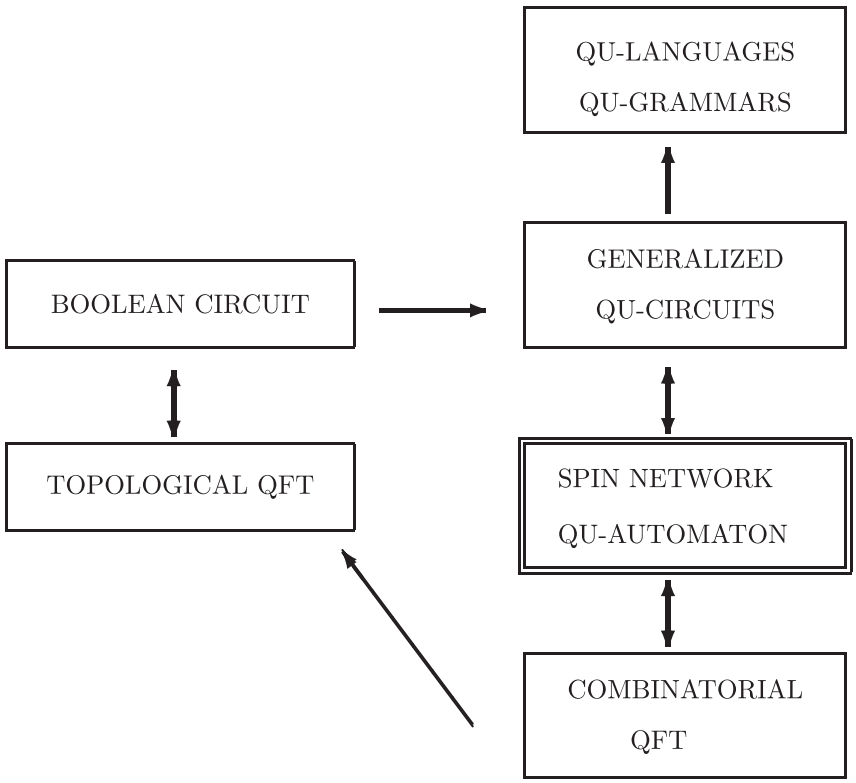
the structure of the model similar to the *state sum models* employed in discretized Topological Quantum Field Theories and Quantum Gravity. The SNQA can thus be viewed as well as a Combinatorial Quantum Field Theoretical model of computation. Most interesting is the application to the algorithmic question related to the topology of knots. Knots can be classified by knot invariants; the farthest reaching of which is the *Jones polynomial*, especially interesting as it is associated with the observables in topological quantum field theory. The problem of computing the Jones polynomial is intractable in the framework of classical complexity theory, yet a quantum algorithm based on the SNQA scheme is capable – as we briefly show here – to approximate it at any arbitrary precision in polynomial time. Because computing Jones' polynomial for any knot is a universal problem, namely the hardest problem that a quantum computer can efficiently handle, this application gives a measure of the reach of the SNQA scheme.

We may summarize the content and perspectives of our discussion in the following diagram, where the SNQA is shown both as a generalized quantum circuit (Section 2) or even a quantum simulator, and as a Combinatorial Quantum Field Theoretical model for computation (Section 3). The standard Boolean quantum circuit is a particular case of this general scheme for computation. The fact that the combinatorial approach can be mapped into the purely topological one completes the picture. Quantum automata are quite promising to address issues such as quantum languages and grammars, quantum encoding, and quantum complexity classes of algorithms, naturally related here to enumerative combinatorics of graphs.



2. GENERALIZED QUANTUM CIRCUITS OUT OF THE ANGULAR MOMENTUM ALGEBRA

We discuss here first some aspects of a novel setting for quantum computation proposed by the authors a few years ago [1–3] and based on the (re)coupling theory of $SU(2)$ angular momenta [4, 5], that we refer to as the *spin network simulator*. This can be thought of as a non-Boolean generalization of the standard quantum circuit model [6], with unitary gates expressed in terms of

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- **(i)** recoupling coefficients ($3nj$ symbols) between inequivalent binary coupling schemes of $N \equiv (n+1)$ $SU(2)$-angular momenta (j-gates);
- **(ii)** Wigner rotations in the eigenspace of the total angular momentum (M-gates).

The picture contains the Boolean case as the particular case when all N angular momenta are spin $\frac{1}{2}$, namely qu-bits. The combinatorial structure of the model closely resembles $SU(2)$ state sum models employed in discretized approaches to 2 and 3-dimensional Topological Quantum Field Theories [7] and Quantum Gravity [8, 9]. An explicit mapping relating the spin network with the topological approach to quantum computation based on modular functors, proposed originally in [10], can be established as done in Ref. 2, section 6 (see also [11] for an overview). Significant developments of the quantum spin network simulator scheme, referred to as *spin network quantum automata*, able to sustain algorithms for computing topological invariants, are then addressed in Section 3.

The basic ingredients of the kinematics of the simulator are computational Hilbert spaces, to be defined in the succeeding text, equipped with all unitary gates of types **(i)** and **(ii)** that we may consistently apply to transform vectors belonging to such spaces. As in a customary circuit – be it classical o quantum – a specific computation is implemented by picking up an input state to generate the corresponding output by acting with a particular (finite) sequence of elementary gates, each performed in one unit of the intrinsic discrete-time of the simulator.

The architecture of the spin network scheme is modeled as a *fiber-bundle* $\mathcal{F}$. $\mathcal{F}$ consists of a base space, each point of which represents an accessible (computational) state of the system; at each point of such space is attached another space, a *fiber*, to describe ('represent') the system internal degrees of freedom. Fiber bundles are necessary to properly describe the *global* (as opposed to *local*) topological properties of the space state. In present case, the fiber is determined by the physics of the system: N angular momenta with fixed (and conserved) sum $\mathbf{J}$. In other words, $\mathcal{F}$ has an $SU(2)$ fiber space structure over a discrete base space V

$$\mathcal{F} \sim (V,\, \mathbb{C}^{2J+1},\, SU(2)^J)_n\,, \tag{1}$$

which encodes all possible computational Hilbert spaces as well as gates for any fixed number $N = n+1$ of incoming angular momenta.

2.1. Computational spaces

The base space $V \doteq \{v(\mathfrak{b})\}$ can be identified with the vertex set of a regular $2(n-1)$-valent graph $\mathfrak{G}_n(V, E)$, which has $2(n-1)$ edge ends converging at each vertex, and cardinality $|V|$ of the vertex set V given by $|V| = (2n-1)!!$ (the double factorial), $\mathfrak{b}$ denoting a specific possible recoupling scheme. There exists a one-to-one correspondence:

$$\{v(\mathfrak{b})\} \longleftrightarrow \left\{\mathcal{H}_n^J(\mathfrak{b})\right\} \tag{2}$$

between the vertices of $\mathfrak{G}_n(V, E)$ and the computational Hilbert spaces of the simulator. For a fixed n, $\mathcal{H}_n^J(\mathfrak{b})$ is the simultaneous eigenspace of the squares of $2n+1$ Hermitean, mutually commuting angular momentum operators, namely

$$\mathbf{J}_1,\, \mathbf{J}_2,\, \mathbf{J}_3, \ldots, \mathbf{J}_{n+1} \equiv \{\mathbf{J}_i\}; \quad \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \ldots + \mathbf{J}_{n+1} \doteq \mathbf{J}\,;$$

$$\mathbf{K}_1,\, \mathbf{K}_2,\, \mathbf{K}_3,\, \ldots,\, \mathbf{K}_{n-1} \equiv \{\mathbf{K}_h\} \tag{3}$$

and of the operator J_z (the projection of the total angular momentum $\mathbf{J}$ along the quantization axis). The operators $\mathbf{K}_h$'s represent intermediate angular momenta. The associated quantum numbers are $j_1, j_2, \ldots, j_{n+1}$; J; $k_1, k_2, \ldots, k_{n-1}$ and M, where $-J \leq M \leq$ in integer steps. If $\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} \otimes \cdots \otimes \mathcal{H}^{j_n} \otimes \mathcal{H}^{j_{n+1}}$ denotes the factorized Hilbert space, namely the $(n+1)$-fold tensor product of the individual irreducible eigenspaces of the $(\mathbf{J}_i)^2$'s, the intermediate angular momenta $\mathbf{K}_h$ are generated through Clebsch–Gordan series, whenever a pair of $\mathbf{J}_i$'s are (binarily) coupled. As an example, by coupling

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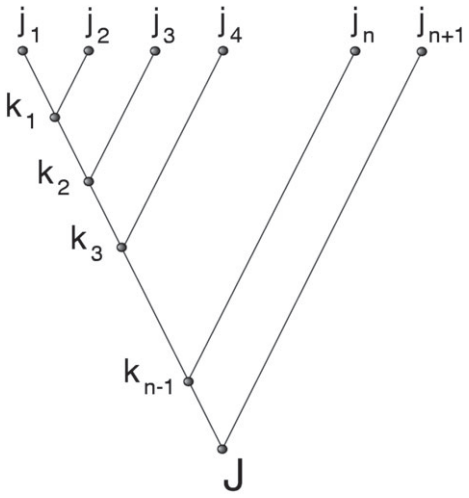

Figure 1. The computational Hilbert spaces are in one-to-one correspondence with rooted binary trees on $(n+1)$ leaves.

sequentially the $\mathbf{J}_i$'s according to the scheme $(\cdots((\mathbf{J}_1+\mathbf{J}_2)+\mathbf{J}_3)+\cdots+\mathbf{J}_{n+1})=\mathbf{J}$ – which generates $(\mathbf{J}_1+\mathbf{J}_2)=\mathbf{K}_1$, $(\mathbf{K}_1+\mathbf{J}_3)=\mathbf{K}_2$, and so on as depicted in Figure 1 – we should get a binary bracketing structure of the type $(\cdots(((\mathcal{H}^{j_1}\otimes\mathcal{H}^{j_2})_{k_1}\otimes\mathcal{H}^{j_3})_{k_2}\otimes\cdots\otimes\mathcal{H}^{j_{n+1}})_{k_{n-1}})_J$, where we add an overall bracket labeled by the quantum number of the total angular momentum J. Note that, as far as j_i's quantum numbers are involved, any value belonging to $\{0, 1/2, 1, 3/2, \ldots\}$ is allowed, while the ranges of the k_h's are suitably constrained by Clebsch–Gordan decompositions (*e.g.* if $(\mathbf{J}_1+\mathbf{J}_2)=\mathbf{K}_1 \Rightarrow |j_1-j_2|\leq k_1\leq j_1+j_2$).

We denote a binary coupled basis of $(n+1)$ angular momenta in the JM-representation and the corresponding Hilbert space appearing in (2) as

$$\{\,|\,[j_1,j_2,j_3,\ldots,j_{n+1}]^{\mathfrak{b}}\,;k_1^{\mathfrak{b}},\,k_2^{\mathfrak{b}},\ldots,k_{n-1}^{\mathfrak{b}}\,;\,JM\,\rangle,\;-J\leq M\leq J\}$$

$$=\;\mathcal{H}_n^J\,(\mathfrak{b})\;\doteq\;\text{span}\,\{\,|\,\mathfrak{b}\,;JM\,\rangle_n\,\}\,, \tag{4}$$

where the string inside $[j_1,j_2,j_3,\ldots,j_{+1}]^{\mathfrak{b}}$ is not necessarily an ordered one, $\mathfrak{b}$ indicates the current binary bracketing structure and the k_h's are uniquely associated with the chain of pairwise couplings selected by $\mathfrak{b}$, *cf.* Figure 1. (Note the Dirac (bra-)ket notation introduced here for quantum states. Such convention is suitable and flexible also with respect to the use of either shorthand or more specific labelings which characterize the states.)

For a given value of J, each $\mathcal{H}_n^J(\mathfrak{b})$ has dimension $(2J+1)$ over $\mathbb{C}$ and thus there exists one isomorphism

$$\mathcal{H}_n^J(\mathfrak{b})\quad\cong_{\mathfrak{b}}\quad\mathbb{C}^{2J+1} \tag{5}$$

for each admissible binary coupling scheme $\mathfrak{b}$ of $(n+1)$ incoming spins. The vector space $\mathbb{C}^{2J+1}$ is interpreted as the typical fiber attached to each vertex $v(\mathfrak{b})\in V$ of the fiber space structure (1) through the isomorphism (5). The fiber is of course given by the moduli space of the total angular momentum group $SU(2)^J$.

2.2. *Unitary gates*

For what concerns unitary operations acting on the computational Hilbert spaces (4), we examine first, as anticipated in **(i)**, the j-gates associated with recoupling coefficients ($3nj$ symbols) of $SU(2)$. It can be shown [4] that any such coefficient can be split into *elementary* j-gates, the Racah transforms, possibly apart from phases and weight factors. A Racah transform applied to a basis vector is defined formally, in the bracketing formalism introduced earlier, as

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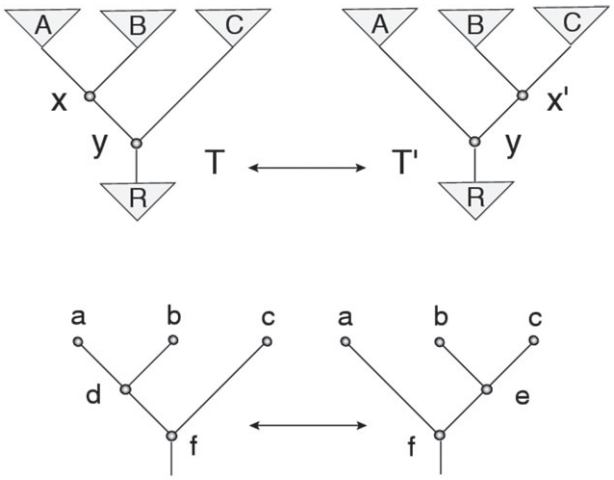

Figure 2. Rotation operations on binary trees can be applied to a general tree (top) made of subtrees A, B, C and R (containing the root), and in particular to the simplest three-leave tree labeled here in such a way as to reproduce the 6j symbol in (7).

$$\mathcal{R} \;:\; |\ldots((a\,b)_d\,c)_f\ldots;JM\rangle \;\mapsto\; |\ldots(a\,(b\,c)_e\,)_f\ldots;JM\rangle, \tag{6}$$

where Latin letters $a, b, c, \ldots$ denote here both incoming (j_i's in the previous notation) and intermediate (k_h's) spin quantum numbers. The explicit expression of (6), graphically represented in Figure 2, reads (skipping dots for a compactified notation)

$$|(a\,(b\,c)_e\,)_f\,;M\rangle = \sum_d (-1)^{a+b+c+f}\,[(2d+1)(2e+1)]^{1/2} \left\{ \begin{array}{ccc} a & b & d \\ c & f & e \end{array} \right\} |((a\,b)_d\,c)_f\,;M\rangle, \tag{7}$$

where there appears the Racah–Wigner 6j symbol of $SU(2)$ (here f plays the role of the total angular momentum quantum number previoulsy denoted J).

Recall that the square of the 6j symbol in (7) gives the probability that a system prepared in the state $|((a\,b)_d\,c)_f\,;M\rangle$ will be measured in the state $|(a\,(b\,c)_e\,)_f\,;M\rangle$. Moreover, Racah transforms $\mathcal{R}$ are the key ingredients to complete the construction of the Rotation graph $\mathfrak{G}_n(V,E)$ introduced earlier (note that in this case, *rotation* refers to a topological operation on binary tree structures and not to the Wigner rotation operators referred to in (**ii**), defined in the succeeding text in Equation (11) and analyzed in details in Appendix A. The edge set $E = \{e\}$ of $\mathfrak{G}_n(V,E)$ is a subset of the Cartesian product $(V \times V)$ selected by the action of elementary j-gates. More precisely, an (undirected) arc between two vertices $v(\mathfrak{b})$ and $v(\mathfrak{b}')$

$$e\,(\mathfrak{b},\mathfrak{b}') \;\doteq\; (v(\mathfrak{b}),\, v(\mathfrak{b}')) \;\in\; (V \times V)\,, \tag{8}$$

exists if, and only if, the underlying Hilbert spaces are related to each other by an elementary unitary operation of the type (6). Note also that elements in E can be considered as mappings

$$\begin{array}{c} (V \times \mathbb{C}^{2J+1})_n \longrightarrow (V \times \mathbb{C}^{2J+1})_n\,, \\ \big(v(\mathfrak{b}),\, \mathcal{H}_n^J(\mathfrak{b})\,\big) \mapsto \big(v(\mathfrak{b}'),\, \mathcal{H}_n^J(\mathfrak{b}')\,\big)\;, \end{array} \tag{9}$$

connecting each given decorated vertex to one of its nearest $2(n-1)$ vertices and thus they define a transport prescription in the horizontal sections belonging to the total space $(V \times \mathbb{C}^{2J+1})_n$ of the fiber bundle (1).

The fundamental feature that characterizes the graph $\mathfrak{G}_n(V,E)$ arises from compatibility conditions satisfied by 6j symbols appearing in (7). The Racah (triangular) identity, the Biedenharn–Elliott (pentagon) identity and the orthogonality conditions for 6j symbols [5] ensure that any simple path in $\mathfrak{G}_n(V,E)$ with fixed endpoints can be freely deformed into any other, providing identical quantum

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transition amplitudes (and of course probabilities) at the kinematical level (*cf.* Ref. 2, section 3.1 for more details).

To complete the description of the structure $(V,\ \mathbb{C}^{2J+1},\ SU(2)^J)_n$ we need to call into play M-gates which act on the angular dependence of vectors in $\mathcal{H}^J_n(\mathfrak{b})$. By expliciting such dependence as

$$\mathcal{H}^J_n(\mathfrak{b}) \ \doteq\ \text{span}\,\{\ |\mathfrak{b};\theta,\ \phi;\ JM\rangle_n\ \}, \tag{10}$$

we write the action of a rotation on a basis vector in the standard form:

$$|\mathfrak{b};\theta',\ \phi';\ M'J\,\rangle_n\ =\ \sum_{M=-J}^{J} D^J_{MM'}\,(\alpha\beta\gamma)\,|\mathfrak{b};\theta,\ \phi;\ JM\,\rangle_n\ , \tag{11}$$

where (θ,ϕ) and (θ',ϕ') are polar angles in the original and rotated coordinate systems, respectively. $D^J_{MM'}\,(\alpha\beta\gamma)$ are Wigner rotation matrices in the JM representation expressed in terms of Euler angles $(\alpha\beta\gamma)$ which form a group under composition [5] (see also Ref. 2, section 3.2 and Appendix B1; and Appendix A in the succeeding text). The shorthand notation $SU(2)^J$ employed in (1) actually refers to the group of W-rotations, which in turn can be interpreted as the automorphism group of the fiber $\mathbb{C}^{2J+1}$. Because rotations in the JM representation do not alter the binary bracketing structure of vectors in computational Hilbert spaces, we can identify the actions of W-matrices as transport prescriptions along the fiber.

The framework outlined shows clearly that we can switch independently j and M-gates without mixing spin and magnetic quantum numbers. This feature, which relies on the discreteness of the base space V and on the triviality of the total space $(V\times\mathbb{C}^{2J+1})_n$, makes it easy with kinematics, although the simulator will exhibit highly non trivial behaviors at the dynamical level, as discussed shortly. A pictorial representation of the computational fiber space $\mathcal{F}(V,\ \mathbb{C}^{2J+1},\ SU(2)^J)_n$ for $(n+1)=4$ incoming spins is sketched in Figure 3. Each vertex is associated, through the mapping (2), with a binary coupled Hilbert space $\mathcal{H}^J_n(\mathfrak{b})$, associated in turn with a binary tree as in Figure 1, and depicted as a space with a few axes. Edges of the graph represent Racah transforms (6) which actually move each space viewed as a whole into one of its nearest $2(n-1)=4$ ones. Inside each $\mathcal{H}^J_n(\mathfrak{b})\cong\mathbb{C}^{2J+1}$ (Equation (5)), we can pick up some particular vector and rotate its components by means of the Hermitean conjugate of the matrix $D^J_{MM'}\,(\alpha\beta\gamma)$ introduced in (11) for some choice of $(\alpha\beta\gamma)$.

Note that the kinematical structure of the spin network complies with all the requisites of an universal quantum simulator as defined by Feynman [12], namely

- *locality*, reflected in the binary bracketing structure of the computational Hilbert spaces, which – together with the action of W-rotations – bears on the existence of local interactions;
- *discreteness of the computational space*, reflected in the combinatorial structure of $\mathfrak{G}_n$;
- *universality*, guaranteed by the properties of the gates described earlier: any unitary transformation operating on computational Hilbert spaces can be reconstructed by taking a finite sequence of Racah transforms (and possibly phases) interlaced with applications of a finite number of W-rotations;
- *discreteness of time*, as assumed in any circuit implementation by simply identifying the time unit τ as the time elapsed between to go from one o the other of any two states connected by an elementary transform.

2.3. Dynamical spin networks

The dynamical behavior of the spin network as a quantum circuit is addressed in terms of directed paths in the fiber space structure $(V,\ \mathbb{C}^{2J+1},\ SU(2)^J)_n$. By a directed path $\mathcal{P}$, we mean a (time) ordered sequence:

$$|\mathfrak{v}_{\text{in}}\,\rangle_n \equiv |\mathfrak{v}_0\,\rangle_n \to |\mathfrak{v}_1\,\rangle_n \to \cdots \to |\mathfrak{v}_s\,\rangle_n \to \cdots \to |\mathfrak{v}_L\,\rangle_n \equiv |\mathfrak{v}_{\text{out}}\,\rangle_n\ , \tag{12}$$

where we use the shorthand notation $|\mathfrak{v}_s\rangle_n$ for computational states and $s=0,1,2,\dots,L$ is the lexicographic labeling of the states along the given path $\mathcal{P}$ with fixed endpoints. L is the length of $\mathcal{P}$, which

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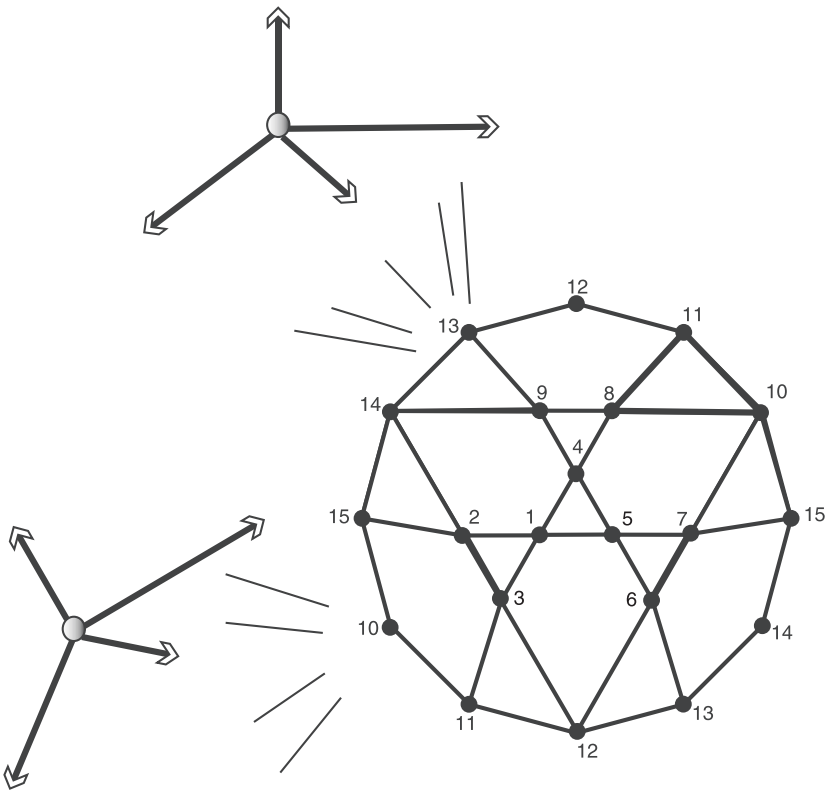

Figure 3. The fiber space structure of the spin network $\mathfrak{G}_3(V,E)$, *that is*, for $(n+1)=4$ incoming spins. The one-to-one correspondence between vertices and binary parenthesization on four letters, say $[a,b,c,d]$, reads: $1 \leftrightarrow (d(b(ac)))$; $2 \leftrightarrow (b(d(ca)))$; $3 \leftrightarrow ((ac)(bd))$; $4 \leftrightarrow (d(a(bc)))$; $5 \leftrightarrow (d(c(ab)))$; $6 \leftrightarrow (c(d(ab)))$; $7 \leftrightarrow ((ab)(cd))$; $8 \leftrightarrow (a(d(bc)))$; $9 \leftrightarrow ((ad)(bc))$; $10 \leftrightarrow (a(b(cd)))$; $11 \leftrightarrow (a(c(bd)))$; $12 \leftrightarrow (c(a(bd)))$; $13 \leftrightarrow (c(b(ad)))$; $14 \leftrightarrow (b(c(da)))$; $15 \leftrightarrow (b(a(cd)))$. Vertices and edges on the perimeter of the graph have to be identified through the antipodal map.

with our convention is proportional to the time duration T of the computation process $L \cdot \tau \doteq T$ in terms of the discrete time unit τ. The integer L characterizing the particular directed path in (12) equals the number of time-ordered elementary operations (computational steps) needed to obtain $|\mathfrak{v}_{\text{out}}\rangle_n$ from $|\mathfrak{v}_{\text{in}}\rangle_n$ following the path $\mathcal{P}$. An elementary computational step, represented by an arrow in (12), is either a Racah transform or a W-rotation: $|\mathfrak{v}_{s+1}\rangle_n = \mathcal{T}_s|\mathfrak{v}_s\rangle_n$, with $\mathcal{T}_s$ either $\mathcal{R}$ as in Equation (6) or a Wigner rotation matrix as in Equation (11).

A circuit-type computation consists in evaluating the quantum expectation value of the unitary operator $\mathfrak{U}_{\mathcal{P}}$ associated with the path $\mathcal{P}$, whereby one obtains the transition probability amplitude (whose square modulus represents the quantum probability of the process)

$$\mathcal{A}_n^{(\mathcal{P})} \doteq_n \langle \mathfrak{v}_{\text{out}} \,|\, \mathfrak{U}_{\mathcal{P}} \,|\, \mathfrak{v}_{\text{in}} \rangle_n \quad ; \quad \left| \mathcal{A}_n^{(\mathcal{P})} \right|^2 = \text{Prob}\left\{ |\mathfrak{v}_{\text{out}}\rangle_n = \mathfrak{U}_{\mathcal{P}} |\mathfrak{v}_{\text{in}}\rangle_n \right\} , \tag{13}$$

where ${}_n\langle \bullet |$ denotes a bra-vector and the ealier shorthand expression means simply that, on applying $\mathfrak{U}_{\mathcal{P}}$ to the input and projecting onto the output, one obtains a measure of which fraction of the output state is generated.

By taking advantage of the possibility of decomposing $\mathfrak{U}_{\mathcal{P}}$ uniquely into an ordered sequence of elementary gates, (13) becomes

$${}_n\langle \mathfrak{v}_{\text{out}} \,|\, \mathfrak{U}_{\mathcal{P}} \,|\, \mathfrak{v}_{\text{in}} \rangle_n \;=\; \lfloor \prod_{s=0}^{L-1} {}_n\langle \mathfrak{v}_{s+1} \,|\, \mathcal{U}_{s,s+1} \,|\, \mathfrak{v}_s \rangle_n \rfloor_{\mathcal{P}} \tag{14}$$

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with $L \equiv L(\mathcal{P})$ for short. The symbol $\lfloor\ \rfloor_{\mathcal{P}}$ indicates that the ordered product along path $\mathcal{P}$ and each elementary operation is now denoted by $\mathcal{U}_{s,s+1}$ ($s = 0, 1, 2, \dots, L(\mathcal{P})-1$) to stress its one-step character with respect to computation. Consequently, each elementary transfer matrix in (14) turns out to be associated with a local Hamiltonian operator $\mathbf{H}_n$ arising from

$$ {}_n\langle \mathfrak{v}_{s+1} \mid \mathcal{U}_{s,s+1} \mid \mathfrak{v}_s \rangle_n \;=\; \exp\{ i\, \mathbf{H}_n(s, s+1) \cdot \tau \} \tag{15} $$

and representing the unitary evolution of the simulator in one unit of its intrinsic time variable τ. We indicate with the shorthand notation $(s, s+1)$ the dependence of $\mathbf{H}_n$ on its variables to make clear the local nature of this operator with respect to the computational space $(V,\ \mathbb{C}^{2J+1},\ SU(2)^J)_n$. When (15) is inserted in (14), such effective Hamiltonians generally do not commute with each other but nonetheless the whole computational process may be mapped to well-defined unitary evolution of the quantum circuit, driven by a poly-local, 2-body-type Hamiltonian, in the internal time interval $T = L(\mathcal{P}) \cdot \tau$.

Different types of evolutions can be grouped into *computing classes* based on the choice of the gates that a particular program (*i.e.*, a collection of directed paths) has to employ. A computing class that alternates j and M-gates would be the most general one (as explained in Ref. 2, section 4.2). However, two particular classes to be illustrated below turn out to be most interesting in view of the possibility of relating them to other models for quantum computation.

The *M-computing class* contains programs which employ only (finite sequences of) M-gates in a suitably chosen computational Hilbert space $\mathcal{H}_n^J(\mathfrak{b})$ and it is not difficult to realize that such kind of computation, when applied to N qu-bits ($\frac{1}{2}$-spins), reproduces the usual Boolean quantum circuit. In Appendix A a detailed analysis of this family is provided.

The *j-computing class* includes programs which employ only j-gates at each computational step. This class is particularly interesting because, as already mentioned, it shares many features with suitable types of discretized TQFTs, the so-called state sum models [8, 9, 11]. Here, the combinatorial structure of Rotation graphs becomes prominent owing to the existence of a one-to-one correspondence between allowed elementary operations and the edge set E of the graph $\mathfrak{G}_n(V, E)$, for any n. We denote the unitary operator associated with a program $\mathcal{P}$ in this class by

$$ \mathcal{U}_{3nj}^{(\mathcal{P})} \;:\; |\mathfrak{v}_{\text{in}}\rangle_n \;\longrightarrow\; |\mathfrak{v}_{\text{out}}\rangle_n \,, \tag{16} $$

where, in a circuit-type setting, input $|\mathfrak{v}_{\text{in}}\rangle_n$ is fixed and $|\mathfrak{v}_{\text{out}}\rangle_n$ is an accepted state. However, in the j-computing class one may address other types of problems. For instance, once selected two states, say $|\mathfrak{v}_{\text{in}}\rangle_n$ and $|\mathfrak{v}_{\text{out}}\rangle_n$, we may consider all possible paths $\mathcal{P}$ that compute $|\mathfrak{v}_{\text{out}}\rangle_n$ as the result of the application of some $\mathcal{U}_{3nj}^{(\mathcal{P})}$ to $|\mathfrak{b}_{\text{in}}\rangle_n$. The functional that takes care of such multiple choices is a *path sum* (a discretized form of Feynman's path integral) which may be written formally as

$$ \mathbf{Z}\left[\, \mathfrak{v}_{\text{in}},\, \mathfrak{v}_{\text{out}} \,\right] \;=\; \sum_{\mathcal{P}} W_{\mathcal{P}}\ {}_n\langle \mathfrak{v}_{\text{out}} \mid \mathcal{U}_{3nj}^{(\mathcal{P})} \mid \mathfrak{v}_{\text{in}} \rangle_n \,, \tag{17} $$

where the summation is over all paths with fixed endpoints and $W_{\mathcal{P}}$ is a weight to be assigned to each particular path. The explicit form of such functional would contain sums over intermediate spin variables k_h's of products of $6j$ symbols with suitable weights (dimensions of $SU(2)$ irreps labeled by k_h's) and phase factors. Note that if we give the same weight, say $W_{\mathcal{P}} = 1$, to each path, then the results on equi-probable amplitudes derived from the algebraic identities for $6j$ symbols ensure that the functional (17) is a combinatorial invariant, namely it is actually independent of the particular path connecting $|\mathfrak{v}_{\text{in}}\rangle_n$ and $|\mathfrak{v}_{\text{out}}\rangle_n$.

In the Boolean qu-bit scheme, the realization of a specific program $\mathcal{P}$ essentially amounts to selecting an appropriate set of $\mathbf{H}_n$, that typically have the form of Heisenberg Hamiltonians; in the present case such Hamiltonians describe instead multi-partite interactions of a finite number of arbitrary angular momenta.

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3. FINITE-STATES Q-DEFORMED AUTOMATA AND PROCESSING OF TOPOLOGICAL INFORMATION

Generally speaking, the search for new algorithmic problems and techniques which would require quantum rather than classical computational resources to be solved is getting more and more challenging. Most quantum algorithms run on the standard quantum circuit model [6], and are designed to solve problems which are essentially number theoretic or decision making. However, other types of problems, typically classified in the field of enumerative combinatorics and ubiquitous in many areas of mathematics and physics, share the feature of being intractable in the framework of classical information theory. These are interesting not only for mathematicians and computer scientists, but are addressed by physicists in the study of both exact solvable models in classical statistical mechanics and quantum field theories whose observables are of geometric nature, such as topological quantum field theories [13].

As an application to such kind of intractable problems, we deal in what follows with the construction of a quantum algorithm for the topological invariants for knots and links. This is a very hard problem that we shall however be able to deal with in terms of spin network quantum automata – to be defined in the succeeding text – that have the advantage of being realizable in terms only of *j*-gates. Knots and braids, besides being fascinating mathematical objects, are encoded into the foundations of a number of physical theories, either as concrete realizations of natural systems or as conceptual tools. A knot is a *knotted* closed curve embedded in the three-dimensional space (links are multi-component knots), commonly presented as a planar diagram, as depicted in Figure 4. As customary in topology, the length of the string and the bending of the various portions of the string itself can be changed at will, provided this is done without cutting and gluing back the endpoints.

Over the years, mathematicians have proposed a number of *knot invariants* aimed to classify systematically all possible knots. Most of these invariants (quantities that depend only on the topological features of the knot) are polynomial expressions (in one or two variables) with coefficients in the relative integers, whose computation belongs to the algorithmic complexity class **#P**. It was Vaughan Jones, who discovered the most famous polynomial invariant, the Jones polynomial [14], which connects knots with quantum field theory and plays a prominent role in the present paper. In the seminal work by Edward Witten [15], the Jones polynomial was actually recognized to be associated with the vacuum expectation value of a so-called Wilson loop operator in a particular type of three-dimensional quantum field theory (the non-Abelian Chern–Simons theory with gauge group $SU(2)$).

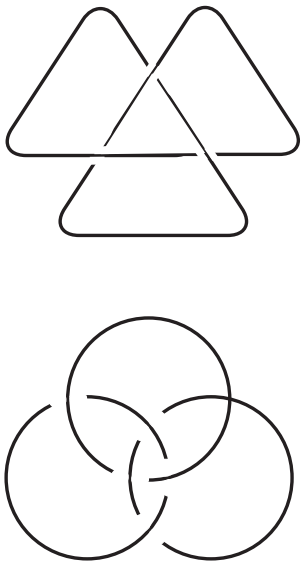

Figure 4. Planar diagrams are projections of knotted curves onto planes taking into account over and undercrossing information. Here, diagrams are shown of the trefoil knot (top) and the Borromean link (bottom).

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Braids – sets made of finite collections of strands with fixed endpoints – appear naturally in this context too, because we can always arrange a knot or a link as the closure of a braid: this crucial property is pictorially illustrated in Figure 5 later. Moreover, braids enrich the purely topological nature of the theory because the set of crossings of any braid can be endowed with a group structure. The Artin braid group on n strands (see [16] and Figure 6) encodes all topological information about 'over' and 'under' crossings into an algebraic setting, opening the possibility of describing polynomial invariants of knots in terms of representations of this group.

The Jones polynomial can be interpreted both as the trace of a (suitably chosen) matrix representation of the braid group [14], and as the vacuum expectation value of an observable in a unitary quantum field theory [15]. In what follows, we restrict our attention to the purely algebraic approach to the definition of *colored Jones polynomials* leaving aside the machinery of quantum field theory, see [17–19] and references therein.

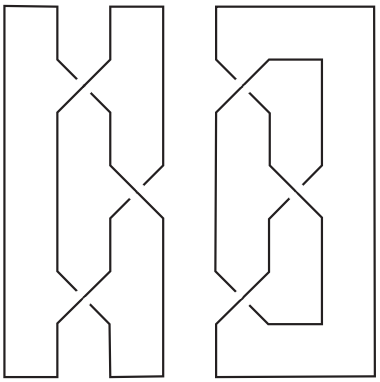

Figure 5. Two presentations of the trefoil knot as closures of open braids.

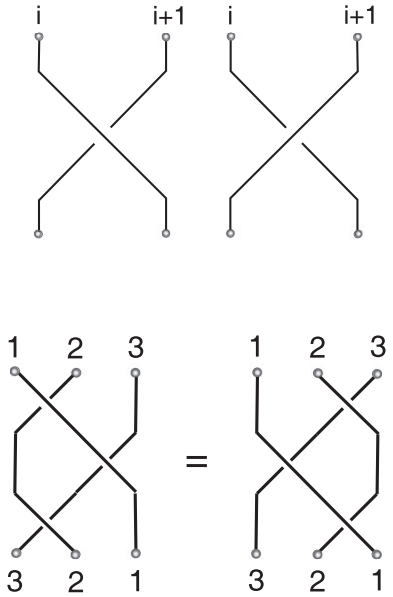

Figure 6. The braid group $\mathbf{B}_n$ is finitely generated in terms of n generators – crossing operations involving two strands – which satisfy suitable conditions. Generator s_i and its inverse s_i^{-1} acting on two contiguous strands (top) and the fundamental algebraic condition relating the action of generators on three strands, $s_1 s_2 s_1 = s_2 s_1 s_2$, (bottom) are illustrated graphically here. The presentation of the group is: $\mathbf{B}_n = \langle s_i, i = 1, \dots, n \,|\, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,,\ i = 1, \dots, n-1\,; s_i s_j = s_j s_i \text{ for } |i-j| \geq 2 \rangle$.

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3.1. Finite-states spin network automata

Looking back at the spin network as the generalized quantum circuit discussed in the previous section, we notice that we are indeed at the presence of an all-purpose machine (which can be built out of molecular, atomic, or nuclear real systems) able to implement in principle any computation. In order to address specific algorithmic problems, we should specify an encoding scheme, choose a particular input state, and design a program; actually select an ordered path $\mathcal{P}$ in the computational space as done in Equation (12). Note that in general such ordered path – a sequence of elementary j and M-gates – is actually a mixed (analogical and discrete) process, and would require a careful preparation of the physical microscopic support in order to keep under control the allowed values of all angles in the rotation W-matrices $D^{J}_{MM'}$, see Equation (11). On the other hand, within a purely j-type computing class, both input/output and the entire path rely on discrete resources which are rotationally invariant by definition and, thanks to algebraic constraints among 6j-symbols, there exist equivalent paths from the input to the output states, all providing the same quantum probability amplitude.

This intriguing view of *quantum parallelism*, reflected into suitable sum over amplitudes of the type introduced earlier in Equation (17), is far-reaching and is deeply embedded within the topological quantum computation scheme [20], which deals with classes of algorithmic problems in geometry and physics that share a global nature; typically the computation of topological invariants of knots.

Before going through some more details, note that 'discrete means' does not imply a finitary information processing when the size of the input (encoded into a string in a finite alphabet) increases. In this respect, the theory of automata and formal languages allows us to address, in particular, what has to be meant by *finite computational processes*, such as those employed in the following for evaluating topological invariants of knots.

If $\mathcal{A}$ is an alphabet, made of letters, digits or other symbols, and $\mathcal{A}^*$ denotes the set of all finite sequences of words over $\mathcal{A}$, a language $\mathcal{L}$ over $\mathcal{A}$ is a subset of $\mathcal{A}^*$. In our notation, the empty word is ϵ, and the concatenation of two words u and v is simply denoted by uv. In the 60s, Noam Chomsky introduced a four level-hierarchy describing formal languages according to their internal structure, namely regular languages, context-free languages, context-sensitive languages, and recursively enumerable languages (recognized by Turing machines). Here, we are interested in finite state-automata, the machines able to accept regular languages, aiming specifically to construct quantum finite-states automata as they can be derived from the spin network setting of Section 2.

Formally, quantum finite states-automata are obtained from their classical probabilistic counterparts by moving from the notion of (classical) probability associated with transitions to quantum probability amplitudes, and computation takes place inside a suitable Hilbert space through unitary operators (matrices). Following [21], the measure-once quantum finite-automaton is defined as a 5-tuple $M = (Q, \Sigma, \delta, \mathbf{q}_0, \mathbf{q}_f)$, where: Q is a finite set of states; Σ is a finite input alphabet with an end-marker symbol #; $\delta : Q \times \Sigma \to Q$ is the transition function; $\delta(\mathbf{q}, \sigma, \mathbf{q}')$ is the probability amplitude for the transition from the state $\mathbf{q}$ to the state $\mathbf{q}'$ upon reading symbol σ; the state $\mathbf{q}_0$ is the initial configuration of the system, and $\mathbf{q}_f$ is the accepted final state. For all states and symbols, the function δ must be unitary. At the end of the computational process, the automaton measures its configuration: if it is in an accepted state then the input is accepted, otherwise it is rejected. The configuration of the automaton is in general a superposition of states in the Hilbert space where the automaton lives. The transition function is represented by a set of unitary matrices $U_\sigma(\sigma \in \Sigma)$, where U_σ represents the unitary transition of the automaton reading the symbol σ. The probability amplitude for the automaton M to accept the string w is finally given by

$$f_M(w) = \left\langle \mathbf{q}_f \right| U_w \left| \mathbf{q}_0 \right\rangle, \tag{18}$$

where it is the explicit form of $f_M(w)$ that defines the language $\mathcal{L}$ accepted by the automaton M. It is quite straightforward to recognize that the spin network computational space $\mathfrak{G}_n(V, E) \times \mathbf{C}^{2J+1}$ (for a fixed n and for any specific choice of J) naturally turns into families of quantum automata.

3.2. Braiding spin networks and representations of $SU(2)_q$

In order to be able to process (closures of) braids, it is necessary to modify the spin network setting in such a way that the information about over or under-crossing of pairs of braid strands might be

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recovered unambiguously through unitary transformations (to be looked at as new types of gates in the computational context). Here, we resort to the algebraic approach embodied into the so-called representation ring $\mathfrak{R}\,(SU(2)_q)$ of the *quantum group* $SU(2)_q$[‡] and defined following in the footsteps of the construction of $SU(2)$-representation theory [4, 5].

The ground ring in which the link invariants will take their values is $\Lambda = \mathbb{Z}[q^{\pm 1}] \subset \mathbb{C}$, with $q = \exp(2\pi i/r)$ a complex primitive r-th root of unity (then in practice these expressions are Laurent polynomials in q and q^{-1} with coefficients in the relative integers $\mathbb{Z}$). The elements of $\mathfrak{R}\,(SU(2)_q)$ are complex Hilbert spaces, invariant under the action of the group (recall that a vector space V is invariant under the action of a group G if $G \times V \to V$, namely transformed vectors keep on belonging to V; such spaces are referred to as invariant G-modules). As it happens for $SU(2)$, it can be shown that $\mathfrak{R}\,(SU(2)_q)$ is spanned by finite-dimensional $SU(2)_q$-modules $\{V^j\}$.

In the case of $SU(2)$, the labels $\{j\}$ (the spin quantum numbers from the quantum mechanical point of view) run over all integers and half-integers $\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$, and each V^j is characterized by its dimension $(2j+1)$ and is irreducible (namely cannot be decomposed into a direct sum of invariant subspaces of lower dimensions). In the q-deformed case, it can be shown that the $SU(2)_q$-modules $\{V^j\}$ are irreducible if and only if the labels $\{j\}$ run over the finite set of *colors* $\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, r\}$. Each V^j, spanned by $(2j+1)$ vectors, can be characterized by a specific scalar $\in \Lambda$, the q-integer or q-dimension $[2j+1]_q$, where $[\mathfrak{n}]_q = (q^{\mathfrak{n}/2} - q^{-\mathfrak{n}/2})/(q^{1/2} - q^{1/2})$ for $\mathfrak{n} \in \mathbb{N}^+$, a positive integer. Thus, for each choice of the integer r, we have a distinguished family of irreducible representations (irreps) of $SU(2)_q$

$$\mathfrak{F}_r = \{\, V^j \,\}_{j=0,\dots,r} \;\; ; \;\; V^j \leftrightarrow [2j+1]_q \,, \tag{19}$$

which makes $\mathfrak{R}\,(SU(2)_q)$ a finitely generated ring. As in the case of $SU(2)$, the ring structure is made explicit in terms of the direct sum $\oplus$ and tensor product $\otimes$ of irreps, namely

$$\begin{aligned} V^j \oplus V^k &\in \mathfrak{R}\,(SU(2)_q) \quad \text{if} \quad j,k \le r \,, \\ V^j \otimes V^k &\in \mathfrak{R}\,(SU(2)_q) \quad \text{if} \quad j+k \le r \,, \end{aligned} \tag{20}$$

where the ranges of the labels have to be suitably restricted with respect to the standard case. The analog of the Clebsch–Gordan series, giving the decomposition of the tensor product of two irreps into a (truncated) direct sum of irreps, reads

$$V^{j_1} \otimes V^{j_2} = \bigoplus_{j=|j_1-j_2|}^{\min\{j_1+j_2,\, r-j_1-j_2\}} V^j. \tag{21}$$

Note however that the ring $\mathfrak{R}(SU(2)_q)$ is much richer than its $'$classical$'$ $SU(2)$ – counterpart because $SU(2)_q$ can be endowed with a *quasi-triangular Hopf algebra* structure. Besides the standard operators $\oplus$ and $\otimes$, we can indeed introduce here a comultiplication $\Delta : SU(2)_q \to SU(2)_q \otimes SU(2)_q$, an antipode map $A : SU(2)_q \to SU(2)_q$, a counit $\varepsilon : SU(2)_q \to \mathbb{C}$ and a distinguished invertible element

$$\mathsf{R} \in SU(2)_q \otimes SU(2)_q, \tag{22}$$

called the *R-matrix*. We do not insist any further on the explicit definitions of Δ (particularly relevant here as it is the operation equivalent – in the new scheme – to the composition of angular momenta in the $SU(2)$ case), A and ε, and refer for instance to [22] for more details.

The far-reaching role played by the R-matrix becomes manifest when we define its action on the tensor product of a pair of irreducible $SU(2)_q$-modules in $\mathfrak{R}(SU(2)_q)$. Denoting by $\hat{\mathsf{R}}$ the operator associated to R, we have

$$\hat{\mathsf{R}} \;:\; V^j \otimes V^k \longrightarrow V^k \otimes V^j, \tag{23}$$

[‡]The 'quantum group' $SU(2)_q$ is not quite a group, rather it is a q-deformation of the universal enveloping algebra (a Hopf algebra) of the Lie algebra of $SU(2)$, with an additional composition law called the *coproduct*, see in the succeeding text some more details.

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where the values of the labels j, k have to be suitably restricted according to (20). These $\hat{\mathsf{R}}$-operators will be referred to as the *braiding operators* associated with the R-matrix (22). If we further extend the action of $\hat{\mathsf{R}}$ to the ordered product of three irreps $V^j \otimes V^k \otimes V^l$ by defining

$$\hat{\mathsf{R}}_{jk} \doteq \hat{\mathsf{R}} \otimes \mathrm{Id} : (V^j \otimes V^k) \otimes V^l \longrightarrow (V^k \otimes V^j) \otimes V^l \,, \tag{24}$$

$$\hat{\mathsf{R}}_{kl} \doteq \mathrm{Id} \otimes \hat{\mathsf{R}} : V^j \otimes (V^k \otimes V^l) \longrightarrow V^j \otimes (V^l \otimes V^k) \,, \tag{25}$$

where Id is the identity operator on the corresponding sub-space, then it can be shown that these operators satisfy the quantum (*i.e.*, q-deformed) Yang–Baxter equation

$$\hat{\mathsf{R}}_{jk}\, \hat{\mathsf{R}}_{kl}\, \hat{\mathsf{R}}_{jk} = \hat{\mathsf{R}}_{kl}\, \hat{\mathsf{R}}_{jk}\, \hat{\mathsf{R}}_{kl} \,. \tag{26}$$

Notice the formal equivalence of this equation with the 'three-body' defining relations of the braid group (see caption of Figure 6). The explicit expression of the braiding operator $\hat{\mathsf{R}}$ (and of its inverse $\hat{\mathsf{R}}^{-1}$) can thus be worked out explicitly by selecting orthonormal basis sets in the $SU(2)_q$-modules V^j, V^k, for each admissible choice of the pair j, k. In such bases, all the braiding operators (23), (24) and (25) are unitary.

With these premises, the 'colored invariant' for an oriented link L with M components can be now consistently interpreted as a single, Λ-linear map

$$\mathrm{J}\,(L; q; j_1, j_2, \ldots, j_M) \;:\; \mathfrak{R}\,(SU(2)_q) \longrightarrow \Lambda \,, \tag{27}$$

where the choice of the integer r in the root of unity q – the running variable of the polynomial – is constrained by the requirement $r \geq M$, at least in the most general case (M distinct colors).

3.3. Quantum automaton computation of Jones polynomial

The step-by-step prescription for working out $\mathrm{J}\,(L; q; j_1, j_2, \ldots, j_M)$ in the simple case case of the Borromean link of Figure 4 (a 3-component knot made of three interlaced rings) depicted in Figure 7 is given in Appendix B.

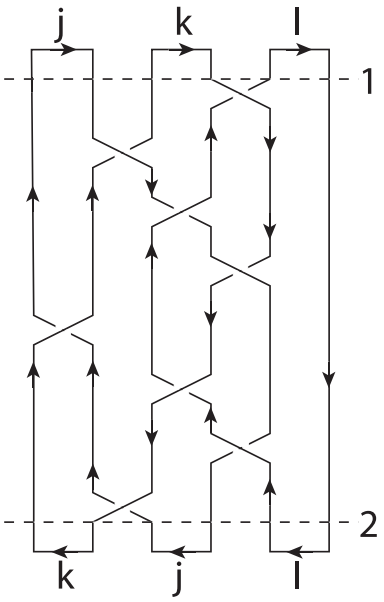

Figure 7. A presentation of the oriented and colored Borromean link as a closed braid on six strands. The parallel straight lines 1 ($\equiv \lambda_1$) and 2 ($\equiv \lambda_2$) intersect the diagram in points to be associated with Hilbert spaces, which inherit the coloring from the corresponding strands.

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Among the $SU(2)_q$-colored link polynomials (27), the Jones invariant [14] is the simplest spin-$\frac{1}{2}$ colored polynomial, and represents the prototype of all of the topological quantum invariants (which include also invariants of 3-dimensional manifolds [22]). The reason why Jones' case is so crucial in the computational context is actually due to the fact that a simpler link invariant that can be computed efficiently, the Alexander–Conway polynomial, is incomplete, while the problem of computing other polynomial invariants is **NP**-hard (see [16, 23] for definitions and reviews on computational questions). Indeed, it was proven in [23] that the evaluation of the Jones polynomial of an alternating link $\tilde{L}$ at a root of unity q is #**P**-hard. 'Alternating' links are special instances of links, the planar diagrams of which exhibit over and under crossings alternatively; thus, the evaluation of the invariant of generic, non-alternating links is at least as hard. The computation becomes feasible when the argument q of the polynomial is a 2nd, 3rd, 4th, and 6th root of unity (refer to [23] for details on this technical issue). Recall finally that the #**P** complexity class can be defined as the class of enumeration problems in which the structures that must be counted are recognizable in polynomial time. A problem π in #**P** is said #**P**-complete if, for any other problem π' in #**P**, π' is polynomial-time reducible to π; if a polynomial time algorithm were found for any such problem, it would follow that #**P** $\subseteq$ **P**. A problem is #**P**-hard if some #**P**-complete problem is polynomial-time reducible to it. Instances of #**P**-complete problems are the counting of Hamiltonian paths in a graph, and the most intractable problems arising in statistical mechanics, such as the enumeration of configurations contributing to ground state partition functions.

Yet, the intractability of the computability of the Jones polynomial does not rule out the possibility of *approximating* efficiently Jones invariant. Given the question *'how hard is it to approximate the Jones polynomials?'*, one has to look at a specific, additive approximation of its value for a fixed q. The latter is a number Z such that, for any choice of a small real $\delta > 0$, the numerical value of $\mathrm{J}(L, q)$, when we substitue in its expression the given value of q, differs from Z by an amount ranging between $-\delta$ and $+\delta$. In a probabilistic setting (either classical or quantum), we require that the value Z can be accepted as an approximation of the polynomial if

$$\text{Prob}\ \left\{ |\,\mathrm{J}(L; q; \{j_i\}) - Z\,| \ \leq \delta \right\} \geq \frac{3}{4}\,, \tag{28}$$

where for simplicity, we consider any link as a closed braid. We refer the reader to [24] and [25] for more accurate statements of (28).

The spin network q-deformed automaton model is based on recoupling transformations, each of which can be decomposed into a series of elementary quantum 6j transformations. The quantum circuit implementing the decomposition of the eight-point fundamental block in terms of $q-6j$ gates is shown in Figure 8, where also the corresponding path on the spin network graph is shown. The spin network

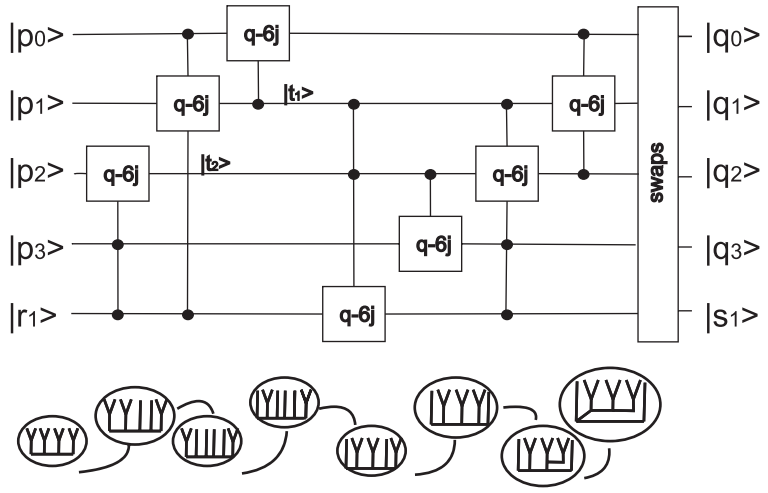

Figure 8. The quantum circuit implementing the decomposition of the eight-point basic ("conformal") block in terms of $q-6j$ gates. An efficient quantum algorithm for the latter has been discussed in [26]. The corresponding path on the spin network graph is shown in the lower part.

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q-deformed automatonhas been shown to be able to implement braiding operators as well as unitary transforms associated with the q-deformed $6j$ and $3nj$ symbols (the q-analog of j-gates in Section 2) in polynomial time with respect to the size of the input [17–19, 27]. The relevant quantities encoding the size of a typical instance of the computational problem – here a link diagram L presented as a closed braid on n strands – are the number of crossings κ of the link and the braid index n. Moreover, each of such elementary unitary operations can be efficiently implemented also in the frame of the standard, Boolean quantum circuit model.

It is worth to recall that further extensions of such q-deformed automaton models have been applied to deal with the efficiency of the calculation of the permanent of a matrix [28] and to model combinatorially the characterization of topological phases of matter and relared anyonic-type computation, see [29, 30] and reference therein.

The deep connection with the q-braided spin-network computational scheme comes out however when one recognizes that the quantity in Equation (B.5) of Appendix B, representing the extended Jones polynomial, is not only the quantum transition amplitude of a finite states-automaton but complies also with the expectation value (14) associated with a path $\mathcal{P}$ in the q version of the spin-network computational space. This does not mean, of course, that we have set up a quantum algorithm for the extended Jones polynomial in the strict sense, because the encoding map could not be efficiently represented (nor approximated) with respect to standard models of computation (Boolean circuits, Turing machines). We provide however instead a quantum system whose evolution can be controlled in such a way that its probability amplitudes give the desired link polynomials.

Finally, let us comment as well on the basic issue characterizing the model of quantum computation adopted. A central role in the scheme is played by the quantum recognizer $\mathcal{R}_{\mathcal{L}}$ (see Appendix C for details) able to process efficiently the language $\mathcal{L}$ generated by the braid group with transition matrices given by Kaul's unitary representations, and the related probability distributions associated with quantum topological invariants. It might be tempting to proceed without this step, processing directly the unitary representations within the quantum circuit scheme of computation. However, the basic morphisms of the q-tensor category on which the recognizer is modeled can be efficiently compiled and approximated on a quantum circuit, as the implementation of a $q-6j$ transformation is independent on both the input size of the algorithmic problem and on the values of its entries. This opens the possibility of looking at the q-spin network simulator as the fundamental model of computation for a wide range of algorithmic problems in geometric topology and group theory.

APPENDIX A: UNITARY M-GATES

M-gates are implemented – inside each computational Hilbert space $\mathcal{H}_n^J$ $(\mathfrak{b})$ – by the Wigner D-functions $D^J_{M\,M'}$, namely the matrix elements of a (unitary) rotation operator $\mathfrak{D}^J_{M\,M'}$ in the JM representation (Equation (11)). A W-matrix can be always expressed as

$$D^J_{M\,M'}\,(\alpha\beta\gamma) \;=\; e^{-iM\alpha}\; d^J_{M\,M'}\,(\beta)\; e^{-iM'\gamma}, \tag{A.1}$$

where $d^J_{M\,M'}\,(\beta)$ is the *reduced* W-matrix. It is worth to recall that in the general case (N any integer and $j_1, j_2, \ldots, j_{n+1}$ chosen in $\{0, 1/2, 1, 3/2, \ldots\}$) the *reducible* $(2J+1)\times(2J+1)$ W-rotation matrix $D^J_{MM'}$ will admit a block diagonal decomposition into irreducible rotation matrices of lower ranks. From the computational point of view, this provides a more general notion of *universal* set of elementary M-gates than that currently adopted in (Boolean) quantum information schemes, typically given in terms of 2×2 (1-qubit) and 4×4 (2-qubit) unitary matrices [6]. Actually, each matrix element of any rotation $\hat{\mathbf{R}}(\alpha,\beta,\gamma)$, parametrized by Euler's angles, $D^J_{MM'} = \langle JM\,|\hat{\mathbf{R}}\,|J'M'\rangle$ can be factorized in a well-defined way, and the procedure is independent of the binary bracketing structure associated to $\mathcal{H}_n^J$ $(\mathfrak{b})$. The explicit expression of such factorization can be written symbolically as

$$\boldsymbol{D}^J\,(\alpha\beta\gamma) \;=\; \sum_{\{m,m'\}}\;\prod_{i=1}^{N}\;\left(\boldsymbol{C}^{k_i}_{k_{i-1}\,j_i}\;\boldsymbol{D}^{j_i}\,(\alpha\beta\gamma)\;\boldsymbol{C}^{k'_i}_{k'_{i-1}\,j_i}\right), \tag{A.2}$$

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where we dropped the matrix indices M, M', m_i, m_i' on the W-matrices $\boldsymbol{D}^J, \boldsymbol{D}^{j_i}$ $(i = 1, 2, \dots, N \equiv n+1)$, and similarly $\boldsymbol{C}^{\bullet}_{\bullet\bullet}$'s are Clebsch–Gordan coefficients with m-type entries omitted. Notice that at the left-hand-side of (A.2) is implicit, and omitted, a factor $\prod_{i=1}^{N} \delta_{k_i,k_i'} \{j_i k_{i-1} k_i\}$, where $\{a\,b\,c\}$ is a symbol that equals 1 if $(a+b+c)$ is an integer and $|a-b| \leq c \leq a+b$; 0 otherwise. The summation is over all magnetic quantum numbers of the angular momentum operators $\{\mathbf{J}_i\}$, while $\{\mathbf{k}_i = \mathbf{j}_1 + \cdots + \mathbf{j}_i\}$ are spin quantum numbers associated with the intermediate operators introduced in (4). In (A.2) there appear N ($\equiv$ # of incoming spins) factors, each containing a W-matrix in the irreducible j_i-th representation of dimension $(2j_i + 1)$, and a total amount of $2N$ Clebsch–Gordan coefficients. Each matrix element of $\boldsymbol{D}^{j_i}$ can be further factorized into the (sum of) product of $2j_i$ W-matrices in the fundamental $j = 1/2$ representation of $SU(2)$. Recall that the explicit form of $D^{\frac{1}{2}}_{m\,m'}$ $(m, m' \in \{\frac{1}{2}, -\frac{1}{2}\})$ reads

$$D^{\frac{1}{2}}_{m\,m'}(\alpha\,\beta\,\gamma) = \begin{pmatrix} e^{-i\alpha/2}\cos(\beta/2)\, e^{-i\gamma/2} & -e^{-i\alpha/2}\sin(\beta/2)\, e^{i\gamma/2} \\ e^{i\alpha/2}\sin(\beta/2)\, e^{-i\gamma/2} & e^{i\alpha/2}\cos(\beta/2)\, e^{i\gamma/2} \end{pmatrix}, \tag{A.3}$$

As a consequence of the aforementioned remarks, we conclude that the elementary factors appearing in the right hand side of (A.2) needed to determine one matrix element of $\boldsymbol{D}^J$, namely $D^J_{M\,M'}$ $(\alpha\,\beta\,\gamma)$ for some MM', are

$$2N \text{ C-G coefficients} \quad ; \quad 2J \equiv 2\sum_{i=1}^{N} j_i \text{ W-matrices } D^{\frac{1}{2}}_{m\,m'} \tag{A.4}$$

and the number of factors one needs in order to evaluate the whole $\boldsymbol{D}^J$ amounts to $[2(J+N)]^{(2J+1)^2}$. These estimates represent in fact upper bounds on the number of factors, because we may reduce the number of elementary W-matrices by employing some 3×3 matrices $\boldsymbol{D}^1$ of the $j = 1$ irrep. Moreover, by considering a purely *fermionic* [*bosonic*] symmetric N-multiplet, the expression (A.2) does not contain C-G coefficients anymore, and the number of elementary factors to be taken into account in (A.4) is simply $2J \equiv N$ [$J \equiv N$, respectively], see Appendix B1 of Ref. [2] and original references therein. Thus the frame of the spin network circuits turns out to be much richer than the Boolean case, as it includes the usual Boolean gates as particular instances. Note finally that $D^{\frac{1}{2}}_{m\,m'}$ is an *elementary* gate in any situation; it is also *universal* for the two particular cases discussed earlier.

APPENDIX B: AUTOMATON PROCESSING OF THE BORROMEAN LINK

The procedure outlined in Section 3 is illustrated here in details.

- Present an M-components link $L = \cup L_i$ $(i = 1, 2, \dots, M)$ as the so-called *plat* closure (Figure 7) of a braid and choose an orientation for each component (depicted by an arrow). Assign to each component a (distinct) 'color' j_i,

$$L_i \quad \longrightarrow \quad j_i \quad (i = 1, 2, \dots, M)\,. \tag{B.1}$$

- Insert two parallel horizontal lines λ_1, λ_2 cutting the *cap* and *cup* portions of the diagram, respectively. This choice provides the diagram with an overall, downward orientation. The region of the diagram lying between λ_1 and λ_2 is an open braid whose strands inherit suitable labels from the colorings (B.1).
- Assign to the intersection point between a line (λ_1 or λ_2) and the string labeled by j the $SU(2)_q$ irreducible module V^j belonging to the distinguished family defined in (19). The whole configurations of intersection points on λ_1 and λ_2, each ordered from left to right, are to be associated with the $SU(2)_q$-modules V_{λ_1} and V_{λ_2}, respectively, each of which is the ordered tensor product of the individual irreps. To give an explicit expression of these correspondences, consider the particular case of the 3-components Borromean link in Figure 7, where

$$\begin{aligned} V_{\lambda_1} &= V^j \otimes V^j \otimes V^k \otimes V^k \otimes V^l \otimes V^l\,, \\ V_{\lambda_2} &= V^k \otimes V^k \otimes V^j \otimes V^j \otimes V^l \otimes V^l\,. \end{aligned} \tag{B.2}$$

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Note that V_{λ_1} and V_{λ_2} have the same dimension as Hilbert spaces over $\mathbb{C}$, given by the product of the dimensions of the individual factors. The number of such factors, say $2N$, is the same for the two spaces and equals the number of strands of the braid, or even the number of caps (cups) lying above the line λ_1 (below λ_2) divided by two. This feature derives of course from the topological presentation we adopted for the link L, because the braid obtained from the plat closure of any link has an even number of strands.

- Going on with the example, in the representation ring $\mathfrak{R}(SU(2)_q)$ there exists the well defined, unitary operator $\hat{\mathrm{B}}(L;q;j,k,l)$ to be associated with the trasformation relating V_{λ_1} and V_{λ_2} in the diagram of the Borromean link L in Figure 7

$$\hat{\mathrm{B}}(L;q;j,k,l) \ : \ V_{\lambda_1} \longrightarrow V_{\lambda_2}, \tag{B.3}$$

where V_{λ_1} and V_{λ_2} are explicitly defined in (B.2). The composite braiding operator $\hat{\mathrm{B}}(L;q;j,k,l)$ can be decomposed into an ordered sequence of the *elementary* unitary braiding operators $\hat{\mathsf{R}}$ (and their inverses) introduced in (23), suitably tensorized with identities. The sequence is uniquely determined by going trough the diagram from λ_1 to λ_2.

- In the case of the Borromean link, the matrix elements of the braiding operator (B.3) evaluated on (the tensor product of) orthonormal basis vectors of the spaces V^j, V^k, V^l can be collected into a unitary $(2J+1)\times(2J+1)$ matrix parametrized by the colors j,k,l, namely

$$\mathrm{B}_{\alpha\beta}(j,k,l) \ \in \ U(\Lambda,\ 2J+1) \ \ (\alpha,\beta = 1,2,\ldots,2J+1), \tag{B.4}$$

where $U(\Lambda,\ 2J+1)$ is the algebra of unitary matrices on the ground ring $\Lambda \equiv \mathbb{Z}[q,q^{-1}]$ and $(2J+1) = (2j+1)(2k+1)(2l+1)$.

Finally, the colored link invariant $\mathrm{J}(L;q;j,k,l)$ is obtained by taking the trace of the matrix (B.4), formally

$$\mathrm{J}(L;q;j,k,l) \ = \ (\mathrm{Tr}\,\mathrm{B}_{\alpha\beta})(j,k,l), \tag{B.5}$$

where the resulting quantity turns out to contain powers of q and q^{-1} and the colorings through the quantum dimensions $[2j+1]_q$, $[2k+1]_q$ and $[2l+1]_q$.

APPENDIX C: THE QUANTUM RECOGNIZER

In the frame of the quantum spin network simulator scheme, a computation is represented by a collection of step-by-step transition rules selecting (families of) 'directed paths' in the spin-network computational space $\mathfrak{G}_n(V,E)\times\mathbb{C}^{2J+1}$; all starting from the same input state and ending in an admissible output state. A single path is thus posed into a one-to-one correspondence with a (finite-state) quantum automaton calculation once a particular encoding scheme is selected for the problem to be addressed.

In the $SU(2)_q$ case (the SNQA), the simulator (or, better, the associated automaton) can be identified as a *quantum recognizer*, a quantum machine that can operate on quantum languages recognizing the family they belong to. First defined by Wiesner and Crutchfield [31] a quantum recognizer $\mathcal{R}_{\mathcal{L}}$ is a particular type of finite-state quantum machine defined as a five-tuple $\{Q,\mathcal{H},X,Y,\mathbb{T}(Y|X)\}$, where: (i) Q is a set of $\mathfrak{n}$ basis states, the internal states of the machine; (ii) $\mathcal{H}$ is an $\mathfrak{n}$-dimensional Hilbert space in which a particular (normalized) state, $|\Psi_0\rangle$, $\in\mathcal{H}$ is singled out as *start state* expressed in the basis Q; (iii) $X,Y\in\{\mathfrak{a},\mathfrak{r},\epsilon\}$ ($\mathfrak{a}\doteq$ **accept**, $\mathfrak{r}\doteq$ **reject**, $\epsilon\doteq$ the **null** symbol) are finite alphabets for input and output symbols respectively; and (iv) $\mathbb{T}(Y|X)$ is the subset of $\mathfrak{n}\times\mathfrak{n}$ transition matrices of the form $\{\ \mathbf{T}(y|x) \ = \ \mathcal{U}(x)\mathbb{P}(y);\ x\in X,\ y\in Y\}$, where $\mathcal{U}(x)$ is a unitary matrix which determines the state vector evolution and $\mathbb{P}(y)$ is a projection operator associated with the output measurement on a complete set of observables associated with the upgraded state vector.

In this kind of machine, the output alphabet is chosen in such a way that a word w written in the input alphabeth X must be either accepted or rejected, while for the null symbol the requirement is $\mathbb{P}(\epsilon)\equiv\mathbb{I}$

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(the identity matrix). The one-step transition matrices applied to the start state $|\Psi_0\rangle$ assume therefore the form, $\forall x \in X$,

$$\mathbf{T}\left(\begin{array}{c|c} \epsilon & \\ \mathfrak{a} & x \\ \mathfrak{r} & \end{array}\right) = \mathcal{U}(x)\begin{pmatrix} \mathbb{P}(\epsilon) \\ \mathbb{P}(\mathfrak{a}) \\ \mathbb{P}(\mathfrak{r}) \end{pmatrix} \equiv \mathcal{U}(x)\begin{pmatrix} \mathbb{I} \\ |\Psi_0\rangle\langle\Psi_0| \\ \mathbb{I} - |\Psi_0\rangle\langle\Psi_0| \end{pmatrix},$$

according to whether no measure is performed or the output is 'accepted' or 'rejected'.

These general axioms can be adapted to make such a machine able to recognize a language $\mathcal{L}$ endowed with a word-probability distribution $\mathfrak{p}(w)$ over the set of words $\{w\} \in \mathcal{L}$. In particular, for any word $w = x_1 x_2 \ldots x_l \in \mathcal{L}$, the recognizer one-step transition matrix elements are required to be of the form $\mathbf{T}_{ij}(x_s) = \mathcal{U}_{ij}(x_s)$ on reading each individual symbol $x_s \in w$, namely no measurement is performed at the intermediate steps (with i, j running from 1 to $\mathfrak{n}$). Each $\mathcal{U}_{ij}(x_s)$ satisfies the condition $|\,\mathcal{U}_{ij}(x_s)\,|^2 > 0$ and the recognizer upgrades the start state to $\mathcal{U}(w)\,|\,\Psi_0\rangle \doteq \mathcal{U}(x_l)\cdots\mathcal{U}(x_1)\,|\,\Psi_0\rangle$. Only then the machine assigns to the word w the number $\mathfrak{p}(w)$, $0 \leq \mathfrak{p}(w) \leq 1$, $\mathfrak{p}(w) = |\,\langle\Psi_0|\,\mathcal{U}(w)\,\mathbb{P}(\mathfrak{a})\,\mathcal{U}(w)|\Psi_0\rangle\,|$; which is the probability of accepting the word w as a whole. More generally, the machine accepts a word w according to the a priori probability distribution $\Pr(w)$ with a word-probability treshold δ, $0 \leq \delta \leq 1$, if and only if $|\,\Pr(w) - \mathfrak{p}(w)\,| \leq \delta$, $\forall w \in \mathcal{L}$. If the accuracy δ is set to 0, the two probability distributions Pr and $\mathfrak{p}$ coincide.

The success of the algorithm in evaluating the goodness of an approximation of the Jones polynomial is due to the feature that transition amplitudes for such automata are indeed just Jones polynomials if $\mathcal{L}$ is the language associated with the braid group. The quantum circuit which efficiently simulates the dynamics of these automata, if appropriately controlled and sampled with a set of measurements, not only approximates the knot invariants, but recognizes as well and counts the group identity elements necessary to approach the combing problem. As for the complexity of the corresponding circuit, because the time complexity of the spin network automaton is polynomial in the size of the input, the algorithm that efficiently simulates the automata is expected to provide an efficient estimation for the latter problem as well.

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