Geometry of perturbed Gaussian states and quantum estimation

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# Geometry of perturbed Gaussian states and quantum estimation

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**Abstract**
We address the non-Gaussianity (nG) of states obtained by weakly perturbing
a Gaussian state and investigate the relationships with quantum estimation.
For classical perturbations, i.e. perturbations to eigenvalues, we found that the
nG of the perturbed state may be written as the quantum Fisher information
(QFI) distance minus a term depending on the infinitesimal energy change,
i.e. it provides a lower bound to statistical distinguishability. Upon moving on
isoenergetic surfaces in a neighbourhood of a Gaussian state, nG thus coincides
with a proper distance in the Hilbert space and exactly quantifies the statistical
distinguishability of the perturbations. On the other hand, for perturbations
leaving the covariance matrix unperturbed, we show that nG provides an upper
bound to the QFI. Our results show that the geometry of non-Gaussian states
in the neighbourhood of a Gaussian state is definitely not trivial and cannot be
subsumed by a differential structure. Nevertheless, the analysis of perturbations
to a Gaussian state reveals that nG may be a resource for quantum estimation.
The nG of specific families of perturbed Gaussian states is analysed in some
detail with the aim of finding the maximally non-Gaussian state obtainable
from a given Gaussian one.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Non-Gaussianity (nG) is a resource for the implementation of continuous variable quantum
information in bosonic systems [1]. Several schemes to generate non-Gaussian states from
Gaussian ones have been proposed, either based on nonlinear interactions or on conditional

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measurements [2–25]. In many cases the effective nonlinearity is small, and so it is the resulting nG. It is thus of interest to investigate the nG of states in the neighbourhood of a Gaussian state, i.e. the nG of slightly perturbed Gaussian states. Besides the fundamental interest [26], this also provides a way to assess different de-Gaussification mechanisms, as well as nG itself as a resource for quantum estimation. Indeed, in an estimation problem where the variation of a parameter affects the Gaussian character of the involved states, one may expect the amount of nG to play a role in determining the estimation precision.

Quantum estimation deals with situations where one tries to infer the value of a parameter $\lambda$ by measuring a different quantity $X$, which is somehow related to $\lambda$. This often happens in quantum mechanics and quantum information where many quantities of interest, e.g. entanglement [27, 28], do not correspond to a proper observable and should be estimated from the measurement of one or more observable quantities [29]. Given a set $\{\varrho_\lambda\}$ of quantum states parametrized by the value of the quantity of interest, an estimator $\hat{\lambda}$ for $\lambda$ is a real function of the outcomes of the measurements performed on $\varrho_\lambda$. The quantum Cramer–Rao theorem [30–33] establishes a lower bound for the variance $\mathrm{Var}(\lambda)$ of any unbiased estimator, i.e. for the estimation precision, $\mathrm{Var}(\lambda) \geqslant (M H(\lambda))^{-1}$ in terms of the number of measurements $M$ and the so-called quantum Fisher information (QFI), which captures the statistical distinguishability of the states within the set. Indeed, the QFI distance itself is proportional to the Bures distance $d_B^2(\varrho_1, \varrho_2) = 2[1 - F(\varrho_1, \varrho_2)]$, $F(\varrho_1, \varrho_2) = \mathrm{Tr}[\sqrt{\sqrt{\varrho_1}\varrho_2\sqrt{\varrho_2}}]$ being the fidelity between states corresponding to infinitesimally close values of the parameter, i.e. in terms of metrics, $H(\lambda) = 4\,g_\lambda = 2\sum_{nm}(\varrho_n + \varrho_m)^{-1}\,|\langle\psi_m|\partial_\lambda\varrho_\lambda|\psi_n\rangle|^2$ where $d_B^2(\varrho_{\lambda+d\lambda}, \varrho_\lambda) = g_\lambda\,d\lambda^2$, and we have used the eigenbasis $\varrho_\lambda = \sum_n \varrho_n|\psi_n\rangle\langle\psi_n|$.

## 2. Gaussian states and a measure of non-Gaussianity

Let us consider a single-mode bosonic system described by the mode operator $a$ with commutation relations $[a, a^\dagger] = 1$. A quantum state $\varrho$ is fully described by its characteristic function $\chi[\varrho](\lambda) = \mathrm{Tr}[\varrho\,D(\lambda)]$ where $D(\lambda) = \exp\{\lambda a^\dagger - \lambda^* a\}$ is the displacement operator. The canonical operators are given by $q = (a + a^\dagger)/\sqrt{2}$ and $p = (a - a^\dagger)/\sqrt{2}\mathrm{i}$ with commutation relations given by $[q, p] = \mathrm{i}$. Upon introducing the vector $\boldsymbol{R} = (q, p)^T$, the covariance matrix $\boldsymbol{\sigma}_\varrho$ and the vector of mean values $\boldsymbol{X}_\varrho$ of a quantum state $\varrho$ are defined as $\sigma_{kj} = \frac{1}{2}\langle R_k R_j + R_j R_k\rangle_\varrho - \langle R_j\rangle_\varrho\langle R_k\rangle_\varrho$ and $X_j = \langle R_j\rangle_\varrho$, where $\langle O\rangle_\varrho = \mathrm{Tr}[\varrho\,O]$ is the expectation value of the operator $O$ on the state $\varrho$. A quantum state is said to be Gaussian if its characteristic function has a Gaussian form. Once the CM and the vectors of mean values are given, a Gaussian state is fully determined.

The amount of nG $\delta[\varrho]$ of a quantum state $\varrho$ may be quantified by the quantum relative entropy [34] $\mathsf{S}(\varrho\|\tau_\varrho) = \mathrm{Tr}[\varrho(\log\varrho - \log\tau)]$ between $\varrho$ and its reference Gaussian state $\tau_\varrho$, which is a Gaussian state with the same covariance matrix $\boldsymbol{\sigma}$ as $\varrho$. As for its classical counterpart, the Kullback–Leibler divergence, it can be demonstrated that $0 \leqslant \mathsf{S}(\varrho\|\tau) < \infty$ when it is definite, i.e. when $\mathrm{supp}\,\varrho \subseteq \mathrm{supp}\,\tau$. In particular, $\mathsf{S}(\varrho\|\tau_\varrho) = 0$ iff $\varrho \equiv \tau_\varrho$ [35, 36]. Since $\tau_\varrho$ is Gaussian $\mathrm{Tr}[(\tau_\varrho - \varrho)\,\log\tau_\varrho] = 0$, i.e. $\mathsf{S}(\varrho\|\tau_\varrho) = \mathsf{S}(\tau_\varrho) - \mathsf{S}(\varrho)$, we may write $\delta[\varrho] = \mathsf{S}(\tau_\varrho) - \mathsf{S}(\varrho)$ where $\mathsf{S}(\varrho) = -\mathrm{Tr}[\varrho\,\log\varrho]$ is the von Neumann entropy of $\varrho$. Finally, since the von Neumann entropy of a single-mode Gaussian state may be written as $h(\sqrt{\det\boldsymbol{\sigma}_\varrho})$ where $h(x) = (x + \frac{1}{2})\log(x + \frac{1}{2}) - (x - \frac{1}{2})\log(x - \frac{1}{2})$ we have

$$\delta[\varrho] = h\big(\sqrt{\det\boldsymbol{\sigma}_\varrho}\big) - \mathsf{S}(\varrho). \tag{1}$$

A generic single-mode Gaussian state may be written as $\tau = U_S\,\nu\,U_S^\dagger$ where $U_S$ is a symplectic operation, i.e. a unitary $U_S = \exp(-\mathrm{i}H)$ resulting from a Hamiltonian $H$ at most

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quadratic in the field operators, and $\nu$ is a chaotic (maximum entropy) state with $n_\nu = \mathrm{Tr}[\nu\, a^\dagger a]$ average thermal quanta, i.e. $\nu = \sum_k p_k |k\rangle\langle k|$, $p_k = n_\nu^k/(1+n_\nu)^{1+k}$ in the Fock number basis.

## 3. Classical perturbations to a Gaussian state

An infinitesimal perturbation of the eigenvalues $p_k$ of a Gaussian state $\tau$, i.e. $p_k \rightarrow p_k + \mathrm{d}p_k$ results in a perturbed state $\varrho = \sum_k (p_k + \mathrm{d}p_k) U_S |k\rangle\langle k| U_S^\dagger$ which, in general, is no longer Gaussian. Since the nG of a state is invariant under symplectic operations we have $\delta[\varrho] = \delta[\eta]$, where $\eta = U_S^\dagger \varrho\, U_S = \sum_k (p_k + \mathrm{d}p_k)|k\rangle\langle k|$ is diagonal in the Fock basis. The Gaussian reference $\tau_\eta$ of $\eta$ is a thermal state with $n_\eta = n_\nu + \mathrm{d}n = \sum_k k\,(p_k + \mathrm{d}p_k)$ average quanta and the nG may be evaluated upon expanding both terms in $\delta[\eta](=\delta[\varrho])$ up to the second order,

$$\delta[\varrho] = \sum_k \frac{\mathrm{d}p_k^2}{2 p_k} - \frac{\mathrm{d}n^2}{2 n_\nu(1+n_\nu)}. \tag{2}$$

The nG of perturbed states is thus given by the sum of two contributions. The first term is the Fisher information of the probability distribution $\{p_k\}$, which coincides with the classical part of the Bures distance in the Hilbert space. The second term is a negative contribution expressed in terms of the infinitesimal change of the average number of quanta. When travelling on surfaces at constant energy the amount of nG coincides with a proper distance in the Hilbert space and, in this case, it has a geometrical interpretation as the infinitesimal Bures distance. At the same time, since Bures distance is proportional to the QFI one, it expresses the statistical distinguishability of states, and we conclude that moving out from a Gaussian state towards its non-Gaussian neighbours is a resource for estimation purposes. Similar conclusions can be made when comparing families of perturbations $\{\mathrm{d}p\}$ corresponding to the same infinitesimal change of energy $\mathrm{d}n^2$: in this case the different amounts of nG induced by the perturbations are quantified by the Bures distance minus a constant term depending on $\mathrm{d}n^2$ and the initial thermal energy $n_\nu$, i.e. the initial purity $\mu \equiv \mathrm{Tr}[\varrho^2] = \mathrm{Tr}[\tau^2] = (2n_\nu + 1)^{-1}$. We summarize the above statements in the following.

**Theorem 1.** *If $\tau_\lambda$ is a Gaussian state and an infinitesimal variation of the value of $\lambda$ drives it into a state $\varrho_{\lambda+\mathrm{d}\lambda}$ with the same eigenvectors, then the QFI distance is equal to the nG $\delta[\varrho_{\lambda+\mathrm{d}\lambda}]$ plus a term depending both on the infinitesimal variation of energy $\mathrm{d}n^2$ and on the initial purity*

$$H(\lambda)\,\mathrm{d}\lambda^2 = \delta[\varrho_{\lambda+\mathrm{d}\lambda}] + \frac{2\mu^2\,\mathrm{d}n^2}{1-\mu^2}.$$

*In particular, for perturbations that leave the energy unperturbed, the nG of the perturbed state coincides with the QFI distance, whereas, in general, it provides a lower bound.*

### 3.1. Examples of finite perturbations

In order to explore specific directions in the neighbourhood of a Gaussian state $\tau$, let us write the perturbation to the eigenvalues as $\mathrm{d}p_k = \epsilon \mu_k$ where $\{\mu_k\}$ is a given distribution. In this case the nG of the perturbed state is given by

$$\delta[\varrho] = \epsilon^2 \left( \sum_k \frac{(p_k - \mu_k)^2}{2 p_k} - \frac{\Delta n_\mu^2}{2 n_t(1+n_t)} \right) + O(\epsilon^3), \tag{3}$$

where $\Delta n_\mu = \sum_k (p_k - \mu_k)\langle k|a^\dagger a|k\rangle$. Let us now consider the families of states generated by the convex combination $\varrho = (1-\epsilon)\tau + \epsilon \varrho_\mu$ of the Gaussian states $\tau$ with the target state
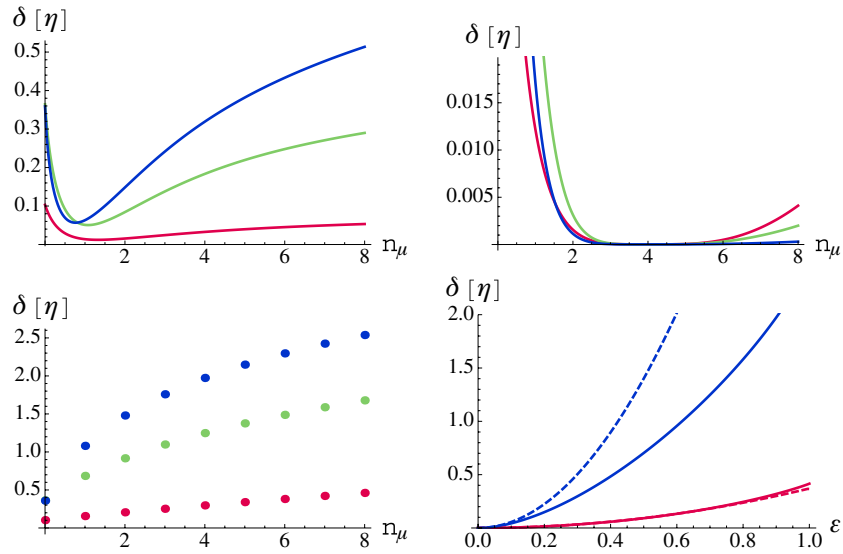
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**Figure 1.** Non-Gaussianity $\delta[\eta]$ as a function of the energy of the target state $n_\mu$, with $n_t = 4$ and for different values of $\epsilon = 0.3, 0.7, 0.9$ (red, green and blue, respectively) for Poissonian target (upper-left panel), thermal (upper right) and Fock (lower left). The lower-right panel shows the nG $\delta[\eta]$ as a function of $\epsilon$, with $n_\mu = n_t = 4$ and for the convex combination with a Poissonian (red) and a Fock (blue) state. Dashed lines correspond to the expansion of nG at the second order in $\epsilon$, as obtained in equation (3) for the corresponding perturbations.

$\varrho_\mu = \sum_k \mu_k U_S |k\rangle\langle k| U_S^\dagger$, which itself is obtained by changing the eigenvalues $p_k$ of the initial Gaussian state to $\mu_k$. Again we exploit invariance of $\delta[\varrho]$ under symplectic operations and focus attention to the diagonal state $\eta = U_S^\dagger \varrho U_S = (1-\epsilon)\eta + \epsilon \eta_\mu$ which has the same nG of $\varrho$. This is the generalization to a finite perturbation of the analysis reported in the previous section, and it is intended as a mean to find the maximally non-Gaussian state obtainable starting from a given Gaussian. The nG of this kind of states can be written as $\delta[\eta] = S(\tau_\eta) - S(\eta) = h[n_\eta + 1/2] - H[q^{(\epsilon)}]$, where $H[q^{(\epsilon)}] = -\sum_k q_k^{(\epsilon)} \log q_k^{(\epsilon)}$ denotes the Shannon entropy of the distribution $q_k^{(\epsilon)} = (1-\epsilon)p_k + \epsilon\mu_k$ and $n_\eta = \text{Tr}[\eta\, a^\dagger a] = (1-\epsilon)n_t + \epsilon n_\mu$ is the average number of quanta of $\eta$. Note that for a thermal state with $n_t$ quanta we have $H[p] = h[n_t + 1/2]$. Using the concavity of the Shannon entropy we obtain an upper bound for the nG

$$\delta[\eta] \leqslant h[n_\eta + 1/2] - h[n_t + 1/2] + \epsilon(h[n_t + 1/2] - H[\mu]). \tag{4}$$

In particular, if the two distributions have the same number of quanta, $n_\mu = n_t$, and thus $n_t = n_\eta$, the bound on $\delta[\eta]$ only depends on the difference between the entropy of the initial and target distributions.

Let us now consider perturbations towards some relevant distributions, i.e. Poissonian $\mu_k^{(p)} = \frac{n_\mu^k}{k!} e^{-n_\mu}$, thermal $\mu_k^{(t)}$, and Fock $\mu_k^{(d)} = \delta_{k,n_\mu}$ and evaluate the nG of states obtained as a convex combination of a thermal state with $n_t$ quanta and a diagonal quantum state with a Poissonian, thermal or Fock distributions and $n_\mu$ quanta.

In figure 1, we plot the nG of the convex combination $\varrho$ as a function of $n_\mu$ for different values of $\epsilon$: if we consider the convex combination with a Fock state, the nG simply increases monotonically with the energy of the added state. For combinations with Poissonian and thermal distributions, we have a maximum for $n_\mu \to 0$; then a local minimum (which for the thermal distribution corresponds trivially to $\delta[\varrho] = 0$) for $n_\mu = n_t$, and then the nG increases

again for higher values of $n_\lambda$. This implies that in order to increase nG the best thing to do is to perturb the initial Gaussian either with a highly excited state $n_\mu \gg 1$ or with the vacuum state $|0\rangle\langle 0|$. If we choose $n_\varrho = n_\mu = n_t$ we know from equation (2) that nG for small perturbations is equivalent to the Bures infinitesimal distance between the two probability distributions $\mu_k$ and $p_k$. In figure 1 (lower-right panel), we show the nG as a function of $\epsilon$: as is apparent from the plot, the nG obtained by adding a Fock state is always much larger than the one for a Poissonian profile. We observe that in the latter case, the expansion at the second order obtained in equation (3) is still accurate for values of $\epsilon$ approaching 1, while it fails to be accurate for $\epsilon \gtrsim 0.2$ for a Fock state, becoming an upper bound on the exact amount of nG. We have also investigated what happens by considering a target distribution randomly chosen on a finite subspace of the infinite Hilbert space: again we have obtained that, at fixed energy of the target state, perturbing with a Fock state yields the biggest increase of nG. This may be easily generalized to a system of $d$ bosonic modes, where the most general Gaussian state is described by $d(2d + 3)$ independent parameters.

## 4. Perturbations at a fixed covariance matrix

Gaussian states are known to be extremal states at a fixed covariance matrix for several relevant quantities, e.g. channel capacities and entanglement measures [37]. Therefore, one may wonder whether perturbing a Gaussian state at a fixed covariance matrix may be quantified in a convenient way for the purposes of quantum estimation. This indeed is the case: the nG provides an upper bound to the QFI distance at a fixed covariance matrix and thus have an operational interpretation in terms of statistical distinguishability. This is more precisely expressed by the following theorem [1].

**Theorem 2.** *If $\tau_\lambda$ is a Gaussian state and an infinitesimal variation of the value of $\lambda$ drives it into a state $\varrho_{\lambda+d\lambda}$ with the same covariance matrix, then the nG $\delta[\varrho_{\lambda+d\lambda}]$ provides an upper bound to the QFI distance*

$$H(\lambda)\,d\lambda^2 \leqslant \delta[\varrho_{\lambda+d\lambda}].$$

**Proof.** If $\varrho_{\lambda+d\lambda}$ and $\tau_\lambda$ have the same CM, then the nG of $\varrho_{\lambda+d\lambda}$, $\delta[\varrho_{\lambda+d\lambda}] = \mathsf{S}(\varrho_{\lambda+d\lambda}||\tau_\lambda) = \widetilde{H}(\lambda)\,d\lambda^2$, where the so-called Kubo–Mori–Bogolubov information $\widetilde{H}(\lambda)$ [32, 38] provides an upper bound for the QFI $H(\lambda) \leqslant \widetilde{H}(\lambda)$ [39], thus proving the theorem.  $\square$

The above theorem says that a larger nG of the perturbed state may correspond to a greater distinguishability from the original one, thus allowing a more precise estimation. Of course, this is not ensured by the theorem, which only provides an upper bound to the QFI. One may wonder that when $\varrho_{\lambda+d\lambda}$ is itself a Gaussian state the theorem requires $H(\lambda) = 0$, i.e. no reliable estimation is possible. Indeed, this should be the case since Gaussian states are uniquely determined by the first two moments, and thus the requirement that the perturbed $\tau_{\lambda+d\lambda}$ and the original state $\tau_\lambda$ are both Gaussian and have the same covariance matrix implies that they are actually the same quantum state.

## 5. Conclusions

In conclusion, we have addressed the nG of states obtained by weakly perturbing Gaussian states and have investigated the relationships with quantum estimation. We found that nG provides a lower bound to the QFI distance for classical perturbations, i.e. perturbations to eigenvalues leaving the eigenvectors unperturbed, and an upper bound for perturbations

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leaving the covariance matrix unperturbed. For situations where the CM is changed by the perturbation, we have no general results. On the other hand, it has already been shown that non-Gaussian states improve quantum estimation of both unitary perturbations as the displacement and the squeezing parameters [40] and nonunitary ones as the loss parameter of a dissipative channel [41]. Overall, our results show that the geometry of non-Gaussian states in the neighbourhood of a Gaussian state is definitely not trivial and cannot be subsumed by a differential structure. Despite this fact, the analysis of perturbations to a Gaussian state may help in revealing when, and to which extent, nG is a resource for quantum estimation. We have also analysed the nG of specific families of perturbed Gaussian states with the aim of finding the maximally non-Gaussian state obtainable from a given Gaussian one.

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## References

[1] Genoni M G and Paris M G A 2010 *Phys. Rev.* A **82** 052341
[2] Olivares S and Paris M G A 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** S616
[3] Kim M S 2008 *J. Phys. B: At. Mol. Opt. Phys.* **41** 133001
[4] Lvovsky A I, Hansen H, Aichele T, Benson O, Mlynek J and Schiller S 2001 *Phys. Rev. Lett.* **87** 050402
[5] Wenger J, Tualle-Brouri R and Grangier P 2004 *Phys. Rev. Lett.* **92** 153601
[6] Zavatta A, Viciani S and Bellini M 2004 *Phys. Rev.* A **70** 053821
[7] Ourjoumtsev A, Tualle-Brouri R, Laurat J and Grangier P 2006 *Science* **312** 83
[8] Neergard-Nielsen J S, Nielsen B M, Hettich C, Molmer K and Polzik E S 2007 *Phys. Rev. Lett.* **97** 083604
[9] Ourjoumtsev A, Jeong H, Tualle-Brouri R and Grangier P 2007 *Nature* **448** 784
[10] Zavatta A, Parigi V and Bellini M 2007 *Phys. Rev.* A **75** 052106
[11] Parigi V, Zavatta A, Kim M S and Bellini M 2007 *Science* **317** 1890
[12] Zavatta A, Parigi V, Kim M S, Jeong H and Bellini M 2009 *Phys. Rev. Lett.* **103** 140406
[13] Ourjoumtsev A, Ferreyrol F, Tualle-Brouri R and Grangier P 2009 *Nature Phys.* **5** 189
[14] Ourjoumtsev A, Dantan A, Tualle-Brouri R and Grangier P 2007 *Phys. Rev. Lett.* **98** 030502
[15] Takahashi H, Neergaard-Nielsen J S, Takeuchi M, Takeoka M, Hayasaka K, Furusawa A and Sasaki M 2010 *Nat. Photonics* **4** 178
[16] Sasaki M and Suzuki S 2006 *Phys. Rev.* A **73** 043807
[17] D'Auria V, de Lisio C, Porzio A, Solimeno S, Anwar J and Paris M G A 2010 *Phys. Rev.* A **81** 033846
[18] Chiummo A, De Laurentis M, Porzio A, Solimeno S and Paris M G A 2005 *Opt. Express* **13** 948
[19] Silberhorn C, Lam P K, Weiß O, König F, Korolkova N and Leuchs G 2001 *Phys. Rev. Lett.* **86** 4267
[20] Glöckl O, Andersen U L and Leuchs G 2006 *Phys. Rev.* A **73** 012306
[21] Tyc T and Korolkova N 2008 *New J. Phys.* **10** 023041
[22] Genoni M, Beduini F A, Allevi A, Bondani M, Olivares S and Paris M G A 2010 *Phys. Scr.* T **140** 014007
[23] Allevi A, Andreoni A, Bondani M, Genoni M G and Olivares S 2010 *Phys. Rev.* A **82** 013816
[24] Allevi A, Andreoni A, Beduini F A, Bondani M, Genoni M G, Olivares S and Paris M G A 2010 *Europhys. Lett.* **92** 20007
[25] Barbieri M, Spagnolo N, Genoni M G, Ferreyrol F, Blandino R, Paris M G A, Grangier P and Tualle-Brouri R 2010 *Phys. Rev.* A **82** 063833
[26] Allegra M, Giorda P and Paris M G A 2010 *Phys. Rev. Lett.* **105** 100503
[27] Genoni M, Giorda P and Paris M G A 2008 *Phys. Rev.* A **78** 032303
[28] Brida G, Degiovanni I, Florio A, Genovese M, Giorda P, Meda A, Paris M G and Shurupov A 2010 *Phys. Rev. Lett.* **104** 100501
[29] Paris M G A 2009 *Int. J. Quantum Inf.* **7** 125
[30] Braunstein S L and Caves C M 1994 *Phys. Rev. Lett.* **72** 3439
     Braunstein S L, Caves C M and Milburn G J 1996 *Ann. Phys.* **247** 135
[31] Fujiwara A 1994 *METR 94-08* (available at www.keisu.t.u-tokyo.ac.jp/research/techrep/data/1994/METR94-09.pdf)

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[32]  Amari S and Nagaoka H 2000 *Methods of Information Geometry (Translations of Mathematical Monographs vol 191)* (Providence, RI: American Mathematical Society)
[33]  Brody D C and Hughston L P 1998 *Proc. R. Soc.* A **454** 2445
      Brody D C and Hughston L P 1999 *Proc. R. Soc.* A **455** 1683
[34]  Genoni M G, Paris M G A and Banaszek K 2008 *Phys. Rev.* A **78** 060303
[35]  Schumacher B and Westmoreland M D 2002 Relative Entropy in Quantum Information Theory *Contemp. Math.* **305** 265–90 (arXiv:quant-ph/0004045)
[36]  Vedral V 2002 *Rev. Mod. Phys.* **74** 197
[37]  Wolf M M, Giedke G and Cirac J I 2006 *Phys. Rev. Lett.* **96** 080502
[38]  Wang X B, Hiroshima T, Tomita A and Hayashi M 2007 *Phys. Rep.* **448** 1
[39]  Petz D 1996 *Linear Algebr. Appl.* **224** 81
[40]  Genoni M G, Invernizzi C and Paris M G A 2009 *Phys. Rev.* A **80** 033842
[41]  Adesso G, Dell'Anno F, Siena S De, Illuminati F and Souza L A M 2009 *Phys. Rev.* A **79** 040305