PHYSICAL REVIEW A **81**, 060101(R) (2010)

Equivalence between XY and dimerized models

Lorenzo Campos Venuti[1] and Marco Roncaglia[2,1]

[1]*Institute for Scientific Interchange, ISI Foundation, Viale S. Severo 65, I-10133 Torino, Italy*

[2]*Dipartimento di Matematica e Informatica, Università degli Studi di Salerno, Via Ponte don Melillo, I-84084 Fisciano (SA), Italy*

(Received 17 February 2010; published 14 June 2010)

The spin-1/2 chain with XY anisotropic coupling in the plane and the XX isotropic dimerized chain are shown to be equivalent in the bulk. For finite systems, we prove that the equivalence is exact in given parity sectors, after taking care of the precise boundary conditions. The proof is given constructively by finding unitary transformations that map the models onto each other. Moreover, we considerably generalized our mapping and showed that even in the case of fully site-dependent couplings the XY chain can be mapped onto an XX model. This result has potential application in the study of disordered systems.

DOI: 10.1103/PhysRevA.81.060101 PACS number(s): 03.65.Ca, 75.10.Pq, 75.10.Nr, 75.10.Jm

I. INTRODUCTION

Exactly solvable models play an important role as limiting cases of more complex system or for testing numerical algorithms. Moreover, their physical properties can generally be calculated exactly and traced back to simple mechanisms that can be used in more complicated scenarios. In this Rapid Communication, we consider two notable solvable models: the anisotropic XY model, originally introduced in [1] with the aim of gaining insights into the long-range properties of the Heisenberg model, and the dimerized XX model, used sometimes as a prototype model to describe spin-Peierls distortion. We prove the equivalence of these two models, although in the literature they are generally considered as separate. The equivalence is shown directly by means of a unitary transformation for their fermionic counterparts and traced back to the spin models, with the boundary conditions carefully taken into account.

For a chain of length L the dimerized XX and anisotropic XY models are given by the following Hamiltonians:

$$H_d^\eta = \frac{1}{2}\sum_{i=1}^{L}[1+\gamma(-1)^i]\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y\right), \qquad (1)$$

$$H_{XY}^\eta = \frac{1}{2}\sum_{i=1}^{L}\left[(1+\gamma)\sigma_i^x\sigma_{i+1}^x+(1-\gamma)\sigma_i^y\sigma_{i+1}^y\right]. \qquad (2)$$

The superscript η denotes different kinds of boundary conditions (BCs), $\sigma_{L+1}^{x,y}=\eta\sigma_1^{x,y}$, $\eta=1,-1,0$, corresponding to periodic (PBCs), antiperiodic (ABCs), and open (OBCs) boundary conditions, respectively. Since both Hamiltonians Eqs. (1) and (2) commute with the parity operator $P=\prod_i\sigma_i^z$, we define the parity sectors $\sigma=\pm1$ and the corresponding projection operators $\Pi_\sigma=(\mathbb{1}+\sigma P)/2$. The central result of this paper establishes that H_d and H_{XY} are unitarily equivalent ($\cong$) up to at most a border term, as precisely stated by the following theorem.

Theorem 1. For L odd and OBCs the models (1) and (2) are unitarily equivalent. For L even and PBCs or ABCs the equivalence holds in given parity blocks depending on the boundary conditions, according to the relation

$$\Pi_\sigma H_d^\eta \Pi_\sigma \cong \Pi_{\eta(-1)^{L/2}} H_{XY}^{\sigma(-1)^{L/2}} \Pi_{\eta(-1)^{L/2}}.$$

In other words, for L even, the boundary index in one model sets the parity sector in the other [times a modulation factor $(-1)^{L/2}$], that is, $\sigma_{XY}=(-1)^{L/2}\eta_d$ and $\sigma_d=(-1)^{L/2}\eta_{XY}$. Note that the two lowest states of the XY model with PBCs, which become exponentially degenerate in L, map onto the ground states of the dimer chain with ABCs or PBCs. An immediate consequence of this theorem is that the two models share the same thermodynamics, since for $L\to\infty$ the effect of boundary terms disappears.

The reason for considering OBCs is partly because of the possibility of using the models Eqs. (1) and (2) to implement quantum information devices. It has been shown in [2] that, in the ground state of the dimer model (though with OBCs and L *even*), the end spins tend to entangle considerably already for small values of the dimerization γ. Moreover, the entanglement survives in the infinite-length limit (long-distance entanglement). In a similar fashion, it was already observed in [1] that the end spins of the anisotropic model order, and such order survives in the thermodynamic limit (TDL). However, this kind of order is of classical nature, and no entanglement is present between the end spins of the open anisotropic chain.[1]

Before proceeding to examine the proof of Theorem 1, let us spend a few words on some benefits of this result. First, let us note that both models commute with π rotations around axes x and y, $\mathcal{R}_\pi^\alpha=\prod_i e^{i\pi\sigma_i^\alpha/2}$, $\alpha=x,y$. However, the dimer model H_d manifests a much larger symmetry; the total magnetization $M^z=\sum_i\sigma_i^z$. This means that H_d is block diagonal in sectors with given magnetization M^z, a feature that is especially useful in the case of nonintegrable extensions of H_d (which maintain this symmetry) where one has to resort to numerical diagonalization. Because of Theorem I, such a symmetry (or an approximate one) must exist also for the anisotropic model H_{XY}. As we will see, the magnetization in the dimer model is mapped onto a nonlocal operator which we are able to compute. Clearly, this operator has the same spectrum as M^z and commutes with H_{XY}.

The proof of Theorem 1 relies on a similar theorem holding for the fermionic version of the models (denoted here with a tilde),

$$\tilde{H}_d^\epsilon = \sum_{i=1}^{L}[1+\gamma(-1)^i](d_i^\dagger d_{i+1}+d_{i+1}^\dagger d_i), \qquad (3)$$

[1]For instance, for $\gamma>0$ the only nonzero correlation surviving in the TDL is $\langle\sigma_1^x\sigma_L^x\rangle$ [1]. Such order is clearly classical.

$$\tilde{H}^{\epsilon}_{XY} = \sum_{i=1}^{L} (a_i^\dagger a_{i+1} + \gamma a_i^\dagger a_{i+1}^\dagger) + \text{H.c.} \tag{4}$$

Here $\epsilon = 1, -1, 0$ distinguishes among PBCs, ABCs, and OBCs for the fermions, that is, $d_{L+1} = \epsilon d_1$ and $a_{L+1} = \epsilon a_1$. As we will see later, the spin systems (1) and (2) are connected to the quadratic fermionic models $\tilde{H}^{\epsilon}_{d}$ and $\tilde{H}^{\epsilon}_{XY}$ via a Jordan-Wigner (JW) transformation, after careful reshuffling of the boundary conditions. The result for the fermionic models is the following theorem.

Theorem 2. In the cases L even and PBCs or ABCs, L odd and OBCs, the models (3) and (4) are unitarily equivalent, that is, there exists a unitary operator U (a "mapping") such that $U \tilde{H}^{\epsilon}_{d} U^\dagger = \tilde{H}^{\epsilon}_{XY}$.

A simple way to recall the different cases in which the theorem applies is given by the following argument. Sending $a_j \to i a_j$ in $\tilde{H}^{\epsilon}_{XY}$, one realizes that the spectrum of $\tilde{H}^{\epsilon}_{XY}$ is invariant under the transformation $\gamma \to -\gamma$. By relabeling the sites of the dimer model, one sees that $\tilde{H}^{\epsilon}_{d}$ possesses the same invariance only when it contains an even number of bonds. This occurs for L even in the case of PBCs or ABCs, and for L odd only in the case of OBCs.

Proof of Theorem 2. Since the fermionic Hamiltonians are quadratic, one way of proving the equivalence between them is to show that they have the same one-body spectrum. To diagonalize the anisotropic model we rewrite the Hamiltonians following the conventions of [1]: $\tilde{H}^{\epsilon}_{XY} = \sum_{i,j} a_i^\dagger A_{i,j} a_j + (1/2)(\sum_{i,j} a_i^\dagger B_{i,j} a_j^\dagger + \text{H.c.})$ and $\tilde{H}^{\epsilon}_{d} = \sum_{i,j} d_i^\dagger M_{ij} d_j$ with matrices given by

$$A = \begin{pmatrix} 0 & 1 & & \epsilon \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ \epsilon & & 1 & 0 \end{pmatrix}, \quad B = \gamma \begin{pmatrix} 0 & 1 & & \epsilon \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ -\epsilon & & -1 & 0 \end{pmatrix},$$

while M is

$$\begin{pmatrix} 0 & 1-\gamma & \cdots & \epsilon[1+(-1)^L\gamma] \\ 1-\gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1-(-1)^L\gamma \\ \epsilon[1+(-1)^L\gamma] & & 1-(-1)^L\gamma & 0 \end{pmatrix}.$$

The one-particle energies of $\tilde{H}^{\epsilon}_{XY}$ are given by the (positive) square root of the eigenvalues of $(A-B)(A+B)$. Calling such roots Λ_k, since A is traceless, one gets [1]

$$\tilde{H}^{\epsilon}_{XY} = \sum_k \Lambda_k \eta_k^\dagger \eta_k - \frac{1}{2} \sum_k \Lambda_k.$$

The equivalence of the two models now stems from the fact that, for L even and PBCs or ABCs, and for L odd and OBCs, $M^2 = (A-B)(A+B)$. Moreover, under the same hypothesis, the eigenvalues of M are symmetric around zero (for L odd and OBCs there is one zero eigenvalue). To write $\tilde{H}^{\epsilon}_{d}$ in the same form as $\tilde{H}^{\epsilon}_{XY}$, perform a particle-hole transformation on the negative eigenvalues of M. We arrive then at $\tilde{H}^{\epsilon}_{d} = \sum_k \Lambda_k \beta_k^\dagger \beta_k - \sum_{\text{neg}} \Lambda_k$, where $\sum_{\text{neg}}$ is the sum over the negative eigenvalues of M. To complete the proof note that, in the specified cases, $\sum_{\text{neg}} \Lambda_k = (1/2) \sum_k \Lambda_k$. ■

II. THE MAPPING

The proof does not give the explicit form of the mapping. We will now provide a physically more compelling proof, which has the additional advantage of revealing an exact form of the mapping. For simplicity we will stick to L even and PBCs or ABCs for the fermionic models. The first step is to write both models in Fourier space,

$$\tilde{H}^{\epsilon}_{d} = \sum_k [2\cos(k) d_k^\dagger d_k + 2i\gamma \sin(k) d_{k+\pi}^\dagger d_k], \tag{5}$$

$$\tilde{H}^{\epsilon}_{XY} = \sum_k \{2\cos(k) a_k^\dagger a_k + \gamma[i \sin(k) a_k^\dagger a_{-k}^\dagger - i \sin(k) a_{-k} a_k]\}. \tag{6}$$

Let us consider first PBCs. The momenta in the Brillouin zone (BZ) are given by $k = 2\pi n/L$, $n = -L/2+1, \ldots, L/2$. Note that, only for PBCs and ABCs, if $k \in \text{BZ}$ then $-k \in \text{BZ}$. Moreover, only for L even, $k \in \text{BZ} \to k + \pi \in \text{BZ}$. In particular, Eqs. (5) and (6) are not correct if L is odd. The unitary transformation that maps the dimer onto the XY model is

$$d_k^\dagger = \begin{cases} a_{-k-\pi}, & -\pi < k < 0, \\ a_k^\dagger, & 0 \leqslant k \leqslant \pi. \end{cases} \tag{7}$$

Notice that the particle-hole transformation does not involve either $k = 0$ or $k = \pi$. In fact, for these two momenta, the dimer model is given by $2(d_0^\dagger d_0 - d_\pi^\dagger d_\pi)$ and the anisotropic one by $2(a_0^\dagger a_0 - a_\pi^\dagger a_\pi)$. The same mapping Eq. (7) transforms $\tilde{H}^{\epsilon}_{d}$ into $\tilde{H}^{\epsilon}_{XY}$ also in the case of ABCs, where the momenta satisfy $k = \pi/L(2n-1)$, $n = -L/2+1, \ldots, L/2$.

The mapping Eq. (7) can be written in a compact form as $d_k^\dagger = f_+(k) a_k^\dagger + f_-(k) a_{-k-\pi}$ with the help of two auxiliary functions $f_\pm(k) := \theta[\pm \sin(k)] \pm \delta_{\sin(k),0}/2$, where θ is the Heaviside function with the convention $\theta(0) = 1/2$.

Because of Eq. (7), the equivalence between (fermionic) dimer and anisotropic models can be generalized. In fact, the mapping transforms an r-nearest-neighbor hopping term into itself, provided r is odd. Instead, an alternating hopping of the form $\sum_i (-1)^i d_i^\dagger d_{i+r} + \text{H.c.}$ becomes $\sum_i (a_i^\dagger a_{i+r}^\dagger + a_{i+r} a_i)$, again for r odd. When r is even the mapping introduces nonanalyticities in Fourier space, and correspondingly the transformed model becomes long ranged in real space. These findings can also be obtained directly by a real-space Fourier back-transform of Eq. (7):

$$d_m^\dagger = \sum_x [\hat{f}_+(m-x) a_x^\dagger + (-1)^x \hat{f}_-(m-x) a_x], \tag{8}$$

with the definition $\hat{f}_\pm(x) = L^{-1} \sum_k e^{-ikx} f_\pm(k)$. If we write simply $\hat{f}_\pm$ in place of the matrix $(\hat{f}_\pm)_{i,j} := \hat{f}(i-j)$, the following relations hold: $\hat{f}_\pm \hat{f}_\pm = \hat{f}_\pm$, and $\hat{f}_+ \hat{f}_- = \hat{f}_- \hat{f}_+ = 0$.

Proof of Theorem 1. The first step is to map the spin models Eqs. (1) and (2) to fermionic models via the JW transformation. In terms of the ladder operator $\sigma_i^\pm = (\sigma_i^x \pm i\sigma^y)/2$, the JW transform is given by $\sigma_i^+ = c_i^\dagger \exp(i\pi \sum_{j=1}^{i-1} c_j^\dagger c_j)$ [this in turn

implies $\sigma_i^- = c_i \exp(-i\pi \sum_{j=1}^{i-1} c_j^\dagger c_j)$, $\sigma_i^z = 2c_i^\dagger c_i - \mathbb{1}$]. The dimer and anisotropic boundary terms become, respectively,

$$H_d^{\eta_d} \to -\eta_d[1+\gamma(-1)^L](d_L^\dagger d_1 + d_1^\dagger d_L)e^{i\pi N_d},$$
$$H_{XY}^{\eta_{XY}} \to -\eta_a(a_L^\dagger a_1 + \gamma a_L^\dagger a_1^\dagger)e^{i\pi N_a} + \text{H.c.},$$

where $N_{d(a)}$ is the total number operator for the d (a) fermions and $\eta_{d(XY)}$ specifies the spin BC for the dimer and XY models. For OBCs, $\eta_d = \eta_{XY} = 0$, we can directly apply the result of Theorem 1 and deduce that the spin models also are unitarily equivalent for L odd. To study the remaining cases, we first need to compute $\exp(i\pi N_d)$ under the action of the mapping Eq. (7). Writing the number operator in Fourier space, we get $N_d = \sum_{0\leqslant k\leqslant\pi} a_k^\dagger a_k + \sum_{-\pi<k<0} a_k a_k^\dagger$. The sum over negative momenta contains a different number of terms depending on the boundary conditions. For PBCs the sum contains $L/2-1$ terms while for ABCs it contains $L/2$ terms. Calling $N_a^+ \equiv \sum_{0\leqslant k\leqslant\pi} a_k^\dagger a_k$ and $N_a^- \equiv \sum_{-\pi<k<0} a_k^\dagger a_k$, we can write compactly $N_d = N_a^+ - N_a^- + L/2 - (1+\epsilon)/2$, where $\epsilon = \pm 1$ defines the boundary conditions of the fermions. Since $N_a^\pm$ are integers, under the action of the mapping we obtain $\exp(i\pi N_d) = -\epsilon(-1)^{L/2}\exp(i\pi N_a)$. Let us now consider the spin models $H_d^{\eta_d}$ ($H_{XY}^{\eta_{XY}}$) in the parity sector σ_d (σ_{XY}). Because of the JW transformation, for L even, the parity operator is $P = e^{i\pi N_{d(a)}}$ and so in each sector $e^{i\pi N_{d(a)}} = \sigma_{d(XY)}$. This means that, in the parity sector σ_d, the spin model with BCs η_d has boundary conditions $-\eta_d\sigma_d$ in the fermions. The same clearly holds for the anisotropic XY model. The equivalence of the fermionic models (Theorem 1) holds when they have the same BCs, which we denote by ϵ. So we arrive at the relation $\eta_d\sigma_d = \eta_a\sigma_a = -\epsilon$. Now we use the mapping of $\exp(i\pi N_d) = \sigma_d$, obtaining $\sigma_d = -\epsilon(-1)^{L/2}\sigma_{XY}$. Solving these last two equations, we finally obtain the parity sectors and boundary conditions under which the equivalence of the spin models applies: $\sigma_{XY} = \eta_d(-1)^{L/2}$ and $\eta_{XY} = \sigma_d(-1)^{L/2}$. ∎

In this proof we have partly seen what happens to the conserved quantity N_d after the action of the mapping. The precise form also depends on the boundary conditions ϵ. In Fourier space we can write $N_d = \sum_k [f_+(k) - f_-(k)]a_k^\dagger a_k + L/2 - (1+\epsilon)/2$. Since the functions $f_\pm(k)$ are not analytic, and by use of $f_+(k) + f_-(k) = 1$, $\forall\, k$, the number operator becomes nonlocal in real space. For example, its explicit form for PBCs ($\epsilon = 1$) is

$$N_d = \frac{2}{L}N_a + \frac{L-2}{2} + \sum_{x\neq y} i^{x-y}\frac{2\sin\left[(x-y)\left(\frac{\pi}{L}+\frac{\pi}{2}\right)\right]}{L\sin\left[(x-y)\frac{\pi}{L}\right]}a_x^\dagger a_y.$$

A corollary of our proof is that this operator commutes with the anisotropic Hamiltonian Eq. (6) ($\epsilon = 1$) and its spectrum is made of integers from zero to L.

III. MAJORANA FERMIONS

The possibility of mapping the dimer model into an anisotropic one is not restricted to the mapping Eq. (7). Another mapping is obtained directly in real space by introduction of Majorana fermions $\zeta_1(j) = (a_j + a_j^\dagger)/\sqrt{2}$, $\zeta_2(j) = i(a_j - a_j^\dagger)/\sqrt{2}$, satisfying $\{\zeta_\alpha(j), \zeta_\beta(j')\} = \delta_{\alpha\beta}\delta_{jj'}$. In this way, it is possible to show that each model gets transformed onto two separate Ising chains in a transverse field, each consisting of $L/2$ sites [3]. Then, assuming L even and PBCs or ABCs, we make the two pairs of Ising chains identical by translating by one site one of the two chains obtained from H_{XY}. The composition of all these steps yields the mapping

$$d_j^\dagger = \tfrac{1}{2}[ia_{j+1}^\dagger + a_j^\dagger - (-1)^j(ia_{j+1} + a_j)]. \tag{9}$$

The transformation above has the advantage of being local in real space and much simpler than Eq. (8). By using Eq. (9) one can reproduce the results of Theorem 1 for PBCs or ABCs. However, Eq. (9) is more powerful in view of its applications to more general local Fermi models. Using the mapping Eq. (9), we can map the "disordered" tight-binding model

$$H_d = \sum_{j=1}^{L} J_j d_j^\dagger d_{j+1} + \text{H.c.} \tag{10}$$

with arbitrary hopping rate J_j onto the generalized anisotropic model

$$H_{XY} = \sum_{j=1}^{L}(J_j^{(+)}a_j^\dagger a_{j+1} + J_j^{(-)}a_j^\dagger a_{j+1}^\dagger) + \text{H.c.}, \tag{11}$$

with $J_j^{(\pm)} = (\pm)^j(J_j \pm J_{j-1})/2$ and J_j always considered periodic, that is, $J_{L+i} = J_i$. This mapping can be further generalized by addition of a uniform and staggered chemical potential. After application of the transformation Eq. (9), such terms become

$$\sum_{j=1}^{L}[\mu + \mu_{\text{st}}(-1)^j]d_j^\dagger d_j - \mu\frac{L}{2}$$
$$= -\frac{i}{2}\sum_{j=1}^{L}[\mu + \mu_{\text{st}}(-1)^j][a_j^\dagger a_{j+1} - (-1)^j a_j^\dagger a_{j+1}^\dagger] + \text{H.c.}$$

The equivalence between the generalized models Eqs. (10) and (11) has potential applications in the study of disordered systems. To obtain results on the random version of the anisotropic model Eq. (11), it is favorable to simulate the Hamiltonian Eq. (10) which conserves the number of excitations. Moreover, through the JW transformation, apart from a possible border term depending on the BCs, the equivalence between Fermi models can be extended to their spin counterparts. In this way a random XY model can be mapped into a random XX model.

IV. CONTINUUM LIMIT

The mappings that we have analyzed so far admit a simple interpretation in the continuum limit. To this end we expand the fermionic fields into chiral components $\psi(x) = e^{ik_F x}R(x) + e^{-ik_F x}L(x)$. For $\gamma = 0$ the two models merge in free massless fermions: $H_0 \equiv \sum_j a_j^\dagger a_{j+1} + \text{H.c.}$ where the band is half filled, so $k_F = \pi/2$. In the continuum limit, we get [4,5]

$$H_0 = i\int_0^L dx[:R^\dagger(x)\partial_x R(x) - L^\dagger(x)\partial_x L(x):] \tag{12}$$

while the mass-generating terms in γ, $\mathcal{O}_{XY} = a_j^\dagger a_{j+1}^\dagger + \text{H.c.}$ and $\mathcal{O}_d = (-1)^j a_j^\dagger a_{j+1} + \text{H.c.}$ become

$$\mathcal{O}_{XY} = i :L^\dagger(x)R^\dagger(x) - R(x)L(x):,$$
$$\mathcal{O}_d = i :L^\dagger(x)R(x) - R^\dagger(x)L(x):.$$

From these expressions, we see directly that the terms multiplied by γ in H_d and H_{XY} are transformed into each other by particle-hole exchange (and a minus sign) on the left movers, $L \to -L^\dagger$, which is reminiscent of the discrete mapping Eq. (7), where the particle-hole transformation was also applied only for negative momenta.

Translating into bosonic language, it is known that the model Eq. (12) is equivalent to the Gaussian model $H_0 = \frac{1}{2}\int dx\{[\partial_x\Theta(x)]^2 + [\partial_x\Phi(x)]^2\}$. The fields Φ and Θ are bosonic and reciprocally dual: $\partial_x\Phi = \partial_\tau\Theta$ and $\partial_\tau\Phi = \partial_x\Theta$. A nonvanishing value of γ has the effect of transforming the Gaussian into the sine-Gordon model by adding a relevant (in the renormalization group sense) term $\mathcal{O}_{XY} = :\sin[\sqrt{4\pi}\Theta(x)]:$ or $\mathcal{O}_d = :\sin[\sqrt{4\pi}\Phi(x)]:$, respectively, in the XY or dimer case. Hence, in the bosonic language, the dimer $\leftrightarrow$ XY mapping simply acts by swapping $\Phi \leftrightarrow \Theta$. It is interesting to observe that a direct consequence of the mapping is the interchange between density and current density, as can be readily inferred from their expressions,

$$\rho(x) = :R^\dagger(x)R(x) + L^\dagger(x)L(x): = -\partial_x\Phi(x)/\sqrt{\pi},$$
$$j(x) = :R^\dagger(x)R(x) - L^\dagger(x)L(x): = \partial_x\Theta(x)/\sqrt{\pi}.$$

Integrating these densities over the space, we obtain two quantum numbers: the total number and the current. In particular, the total number and current are directly related to the two winding numbers $m,n \in \mathbb{Z}$ of, respectively, Θ and Φ. These integers (which determine the scaling dimensions of the primary operators in the Gaussian model) are both good quantum numbers for $\gamma = 0$. For $\gamma \neq 0$, the breaking of translational symmetry in the dimer chain invalidates the conservation of the current, but maintains the particle number conservation. In the XY model the situation is just reversed: the particle number is no longer conserved, owing to the pair creation-destruction terms, while the current remains a good quantum number.

V. HIGHER DIMENSIONS

The mapping described in Eq. (7) can be easily generalized to D dimensions. In a hypercubic D-dimensional lattice the anisotropic model reads $\tilde{H}_{XY} = \sum_{i=1}^D \sum_{\mathbf{x}} (a_{\mathbf{x}}^\dagger a_{\mathbf{x}+\mathbf{e}_i} + \gamma a_{\mathbf{x}}^\dagger a_{\mathbf{x}+\mathbf{e}_i}^\dagger) + \text{H.c.}$, where $\mathbf{x} = (x_1, \ldots, x_D)$ and $\mathbf{e}_i$ is the unit vector along the ith direction. We do not specify BCs here; to fix ideas we can take PBCs on a bipartite lattice. After Fourier transforming, one realizes that the BZ is contained in $[-\pi,\pi]^D$. Now, let us divide the BZ into two regions according to the sign of the first moment k_1: $A = \{\mathbf{k} \in \text{BZ}, : k_1 \in [0,\pi]\}$ and $B = \{\mathbf{k} \in \text{BZ}, : k_1 \in (-\pi,0)\}$. The canonical transformation $a_{\mathbf{k}}^\dagger = d_{\mathbf{k}}^\dagger$ for $\mathbf{k} \in A$ and $a_{\mathbf{k}}^\dagger = d_{-\mathbf{k}-\boldsymbol{\pi}}$ for $\mathbf{k} \in B$ with $\boldsymbol{\pi} = (\pi,\pi,\ldots,\pi)$ generalizes the one-dimensional version Eq. (7). This mapping transforms the XY Hamiltonian into the D-dimensional dimer model:

$$\begin{aligned}\tilde{H}_{XY} &= \sum_{i=1}^D \sum_{\mathbf{k}} \{2\cos(\mathbf{k}\cdot\mathbf{e}_i) d_{\mathbf{k}}^\dagger d_{\mathbf{k}} \\ &\quad + [i\gamma \sin(\mathbf{k}\cdot\mathbf{e}_i) d_{\mathbf{k}}^\dagger d_{\mathbf{k}+\boldsymbol{\pi}} + \text{H.c.}]\} \\ &= \sum_{i=1}^D \sum_{\mathbf{x}} [1 + \gamma(-1)^{|\mathbf{x}|}] d_{\mathbf{x}}^\dagger d_{\mathbf{x}+\mathbf{e}_i} + \text{H.c.},\end{aligned}$$

where the modulation factor is $(-1)^{|\mathbf{x}|} = \exp(i\boldsymbol{\pi}\cdot\mathbf{x})$.

CONCLUSIONS

In this paper, we have analyzed two common spin models (XY and dimerized XX) and shown that they are unitarily equivalent apart from at most a border term. By explicitly providing the unitary transformation, we have been able to generalize the equivalence in many ways. For example, the fully disordered (with site-dependent couplings) XY chain can be mapped onto a disordered XX chain. Considering the fermionic counterpart, we have also shown that generally a dimerized, r-nearest-neighbor hopping term is mapped onto an r-nearest-neighbor pair creation term. In one dimension, our mappings have a simple interpretation in the continuum limit in terms of bosonic fields. Similar considerations can also be extended to higher dimensions.

Mapping XY onto XX models can be useful in view of numerical simulations of disordered models or nonintegrable extensions. This is due to the explicit particle number conservation of the XX models, which makes them easier to treat numerically. A by-product of our analysis is that particle number symmetry is also present in the XY models, although in a hidden fashion.

ACKNOWLEDGMENTS

We are grateful to T. Giamarchi for inspiring us with the picture in the continuum and D. Mattis for reading the manuscript. We also thank J. I. Cirac, and Z. Zimboras for interesting discussions. We have been supported by the EU-STREP Projects HIP (Grant No. 221889) and COQUIT (Grant No. 233747).

[1] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961).

[2] L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia, Phys. Rev. Lett. **96**, 247206 (2006); L. Campos Venuti, S. M. Giampaolo, F. Illuminati, and P. Zanardi, Phys. Rev. A **76**, 052328 (2007); E. I. Kuznetsova and E. B. Fel'dman, J. Exp. Theor. Phys. **102**, 882 (2006).

[3] R. Jullien and J. N. Fields, Phys. Lett. A **69**, 214 (1978); F. Iglói, R. Juhász, and H. Rieger, Phys. Rev. B **61**, 11552 (2000); J. H. H. Perk and H. W. Capel, Physica A **89**, 265 (1977).

[4] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2004).

[5] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, 1998).