

## Coevolution of Glauber-like Ising dynamics and topology

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We study the coevolution of a generalized Glauber dynamics for Ising spins with tunable threshold and of the graph topology where the dynamics takes place. This simple coevolution dynamics generates a rich phase diagram in the space of the two parameters of the model, the threshold and the rewiring probability. The diagram displays phase transitions of different types: spin ordering, percolation, and connectedness. At variance with traditional coevolution models, in which all spins of each connected component of the graph have equal value in the stationary state, we find that, for suitable choices of the parameters, the system may converge to a state in which spins of opposite sign coexist in the same component organized in compact clusters of like-signed spins. Mean field calculations enable one to estimate some features of the phase diagram.

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In recent times there has been an increasing attention, by the statistical physics community, toward applications to social systems and relative phenomena [1]. The goal is the description and possibly the prediction of collective features of processes involving large numbers of individuals without detailed information on the characteristics of the single elements, much like it happens in the physics of phase transitions [2]. Many simple models have been devised, inspired by intuitive ideas on how social interactions between individuals take place. Such models are often variations of known models of statistical physics or entirely new and interesting types of dynamics. The main ingredients are a graph, representing the social network of interactions (acquaintances) between individuals, and a set of local rules, indicating how the state of an agent is affected by (or affects) the state of its neighbors. The graph may be a lattice or have a more complex topology, reflecting properties observed in real social networks [3–5]. Usually one studies the model dynamics on a given graph topology, which remains frozen during the whole evolution of the process. However, in real social phenomena the dynamics of states is often coupled to the transformation of the social network where the process takes place, as the network evolves as well, and the time scales of the two evolutions may be comparable. So, a realistic description of social processes must consider the coevolution of state dynamics and network topology. In the last years several coevolution models have been proposed [6–17]. The interaction rules of such models combine both changes in the states of the agents and in the link structure of the underlying graph. Frozen states of the dynamics are usually characterized by a network composed of one or more connected components with all agents in each component being in the same state. Indeed, the dynamics of states in each component becomes independent of the dynamics ruling the states of the other components and, while the agents of each component converge to the same state, such a state will usually be different from one component to another. In several models both scenarios, i. e. one component with all agents in the same state and two or more separate components each in a different state, can be reached by suitable

choices of the parameter weighing the relative importance of the dynamics of the states versus that of the graph topology [8,9,11,13–15]. Such a scenario is quite simple but it is not very realistic. For example, these models cannot describe the situation in which different groups of people sharing the same state (domains) coexist in the same component, something which is likely to happen in society. In this article, we present the first model that accounts for this situation as well. Our model is based on a simple Glauber-type dynamics for Ising spins [18]. It can also be seen as a sort of threshold model [19] where disorder is in the topology and not in the thresholds. We show that, in spite of its simplicity, the model has a very rich behavior, with several phases, separated by transitions involving both the spin states and the graph topology.

The starting point is a random graph in the manner of Erdős-Rényi [20] with  $N$  nodes and  $M$  links, with  $M = \frac{\langle k \rangle N}{2}$ ,  $\langle k \rangle$  being the average degree of the graph. We stress that the main results do not depend on the initial network topology because the rewiring dynamics leads inevitably to a random network with a Poisson degree distribution. Agents lie on the nodes of the graph, and are endowed with binary states (spins)  $\sigma = \pm 1$ , which are initially assigned at random with equal probability 1/2. The dynamics of the model is defined by iterating the following update rule:

(1) A node  $i$  is selected at random: we indicate with  $k_i$  the number of its neighbors and with  $l_i$  the number of neighbors in the same state.

(2) If  $l_i/k_i \geq s$ , where  $0 \leq s \leq 1$ , the node is stable and nothing happens; otherwise a neighbor  $j$  with  $\sigma_j \neq \sigma_i$  is randomly chosen and: with probability  $\phi$ ,  $i$  cuts its link to  $j$  and attaches it to a randomly chosen node  $l$  such that  $\sigma_l = \sigma_i$  and  $l$  is not already connected to  $i$  [23] (*rewiring*); with probability  $1 - \phi$ ,  $i$  adopts  $j$ 's state (*spin flip*).

The model has two relevant parameters: the threshold  $s$  and the probability of rewiring  $\phi$ . The threshold sets the minimum fraction of neighbors in the same state that a node must have to be stable. In this respect it is a measure of the sensitivity of agents against the social pressure exerted by

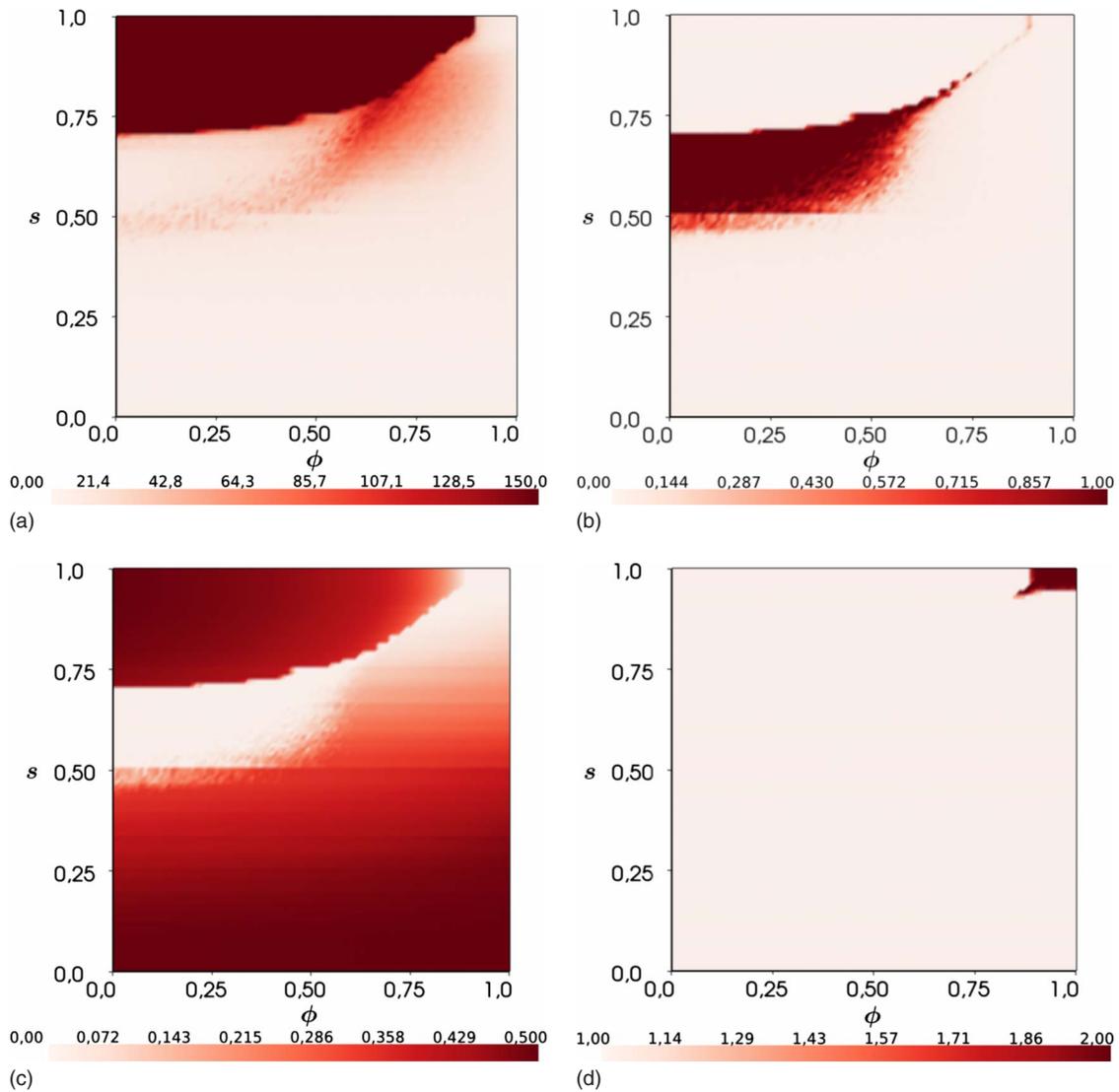


FIG. 1. (Color online) Dependence on the threshold  $s$  and the rewiring probability  $\phi$  of several variables: the convergence time  $t_c$  (top left), the absolute value of the magnetization (top right), the density of unsatisfied links (bottom left) and the number of connected components (bottom right). Data are obtained from numerical simulations on a graph with  $N=50\,000$ ,  $\langle k \rangle=10$ . The darker color in the top left panel means that the convergence time is larger than 150 times the number of nodes. Data in the three other panels are computed after  $1.5 \times 10^6$  iterations.

neighbors with opposite state. If  $s$  is very small virtually all nodes are stable, i. e. they do not flip their spin nor rewire their connections. On the contrary, if  $s$  is close to 1 only nodes fully surrounded by nodes in the same state are stable. When  $s=1/2$ , the spin dynamics is essentially the Glauber dynamics of Ising spins at zero temperature. For  $s < 1/2$  the dynamics is rather uninteresting as already in the initial condition typically nodes have at least half of the neighbors in the same state and hence they are stable. Therefore we focus on the range  $1/2 \leq s \leq 1$ . In general, the presence of a threshold allows for the existence of *unsatisfied links* (i.e., links joining agents with different states) in stable states of the system, at variance with standard coevolution models. The rewiring probability  $\phi$  is a measure of the relative importance of the rate at which the network evolves with respect to the rate at which the state of the node changes. The extreme values correspond to pure spin dynamics on a fixed network

topology ( $\phi=0$ ) and to pure network evolution with no spin dynamics ( $\phi=1$ ).

The phase diagram of the model has a remarkably rich structure. To study its features, we monitor the behavior of some standard observables, the magnetization  $m = \sum_i \sigma_i / N$  and the density of unsatisfied links  $\rho = \sum_{i < j} A_{ij} [1 - \delta(\sigma_i, \sigma_j)] / M$ , where  $A_{ij}$  is the element of the adjacency matrix of the graph ( $A_{ij}=1$ , if  $i$  and  $j$  are neighbors, otherwise  $A_{ij}=0$ ) and  $\delta$  is Kronecker's delta function. Moreover we consider the convergence time  $t_c$ , defined as the time needed to reach a frozen configuration, at which the dynamics stops (i.e.,  $l_i/k_i > s$  for any  $i$ ). In Fig. 1 we report in the plane  $(s, \phi)$  the numerical value of  $t_c$ , as well as the value of  $|m|$  and  $\rho$  and the number of connected components of the graph, after a very long run, sufficient to reach the stationary state. From the behavior of the convergence time, it is clear that the parameter space is divided into two regions. In the

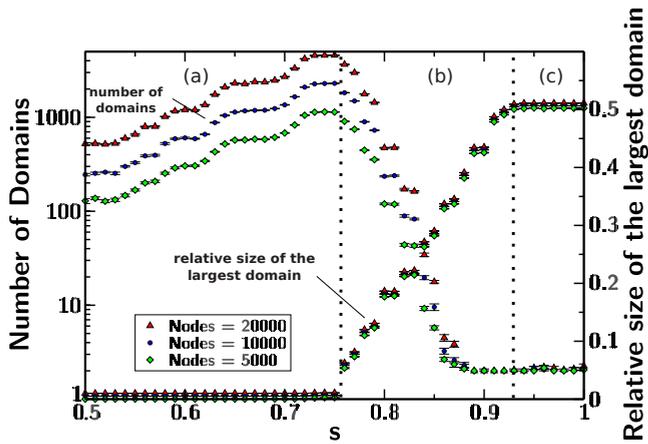


FIG. 2. (Color online) Number of domains and density of the largest domain as a function of  $s$  obtained from simulations on a graph with  $\langle k \rangle = 10$  and  $\phi = 0.95$ .

upper left zone (denoted as  $S-0$ ) there is a phase with ongoing dynamic activity. In this region the convergence time  $t_c$  diverges exponentially with  $N$  so that it is effectively infinite for systems of any reasonable size. However, the system reaches a stationary state, with constant value of the observables. Elsewhere, instead, the dynamics leads in a finite time to an absorbing frozen state with no dynamics. The two regions are separated by an absorbing-state phase transition [21]. The values of  $|m|$  and  $\rho$  indicate that the active phase is disordered: the average magnetization remains zero and the density of unsatisfied links remains high. In this phase, due to the high value of  $s$ , sites are rarely stable and they keep rewiring, looking for similar partners, as in Refs. [11,12]. The structure of the absorbing phase is much richer, as one can identify several distinct subphases, with various types of internal organization. For small values of  $\phi$  and  $s$ , there is a phase (A-I) where  $|m|=1$  and  $\rho=0$ . Here the relatively slow rewiring process allows spin ordering to be completed while the topology remains globally connected. The opposite occurs in the upper right corner of the parameter space (phase A-II). For large  $s$  and  $\phi$ , the value of  $\rho$  is the same of the A-I zone, but in the A-II zone the number of connected components is 2: the system splits into two topologically separated sets of similar size, one of them fully ordered with  $\sigma=1$  and the other with  $\sigma=-1$ . Phases A-I and A-II correspond to those found in the voter model [13]. To understand the organization of the system in the rest of the plane, we study the presence and extent of homogeneous domains, intended as subsets of nodes of the network with two properties: (1) all nodes of the subset are in the same state and (2) any pair of nodes of the subset can be joined with a path within the subset. We then measure two new observables: the number of homogeneous domains in the network and the relative size of the largest of them. In Fig. 2 we plot these two observables as a function of  $s$  for a fixed large value of the rewiring probability  $\phi$ . It is possible to identify two new phases, delimited by two threshold values  $s_p$  and  $s_q$  (indicated by the two vertical dotted lines).

For  $s < s_p$  there are just microscopic domains and their number is proportional to the number of nodes in the net-

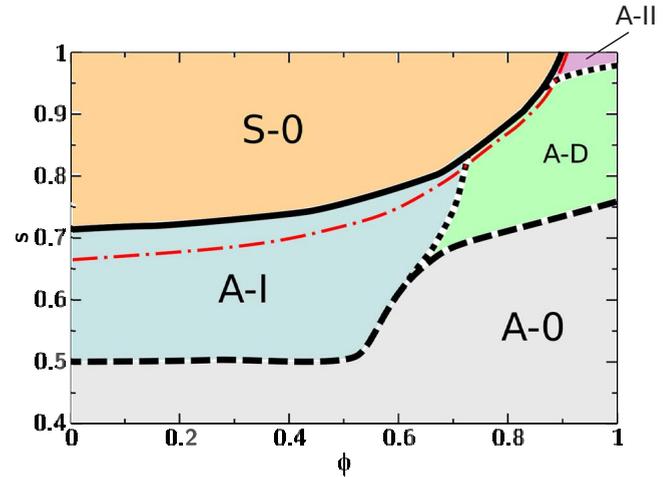


FIG. 3. (Color online) Schematic representation of the phase diagram obtained from numerical simulations on a graph with  $N = 50\,000$ ,  $\langle k \rangle = 10$ , after  $1.5 \times 10^6$  iterations. We can identify two types of phases: the dynamic stationary phase (denoted by  $S-0$ ) and four absorbing phases (denoted by A), divided by the black solid line. Each absorbing phase is characterized by a different organization of the network in domains (see text for details). The (red) dot-dashed line corresponds to the numerical solution of mean-field equations.

work. In this phase (A-0) spanning the whole range of  $\phi$  for  $s < 1/2$ ,  $m=0$ , and  $\rho \sim 1/2$ . Stability is rather easy to reach for all nodes, after few spin flips or link rewirings. We stress that A-0 and S-0 are different: in the case of A-0 an absorbing state is always reached, while for S-0 the system reaches only a dynamic stationary state. At  $s=s_p$  a percolation transition [22] takes place: for  $s_p < s < s_q$  the dynamics lasts long enough to allow for the formation of macroscopic domains (typically two) that grow bigger as  $s$  is increased up to the point where they occupy the whole system (phase A-D). Finally, for still higher values of the threshold  $s > s_q$  the macroscopic domains become topologically disconnected from each other and coincide with the two connected components of the network (phase A-II).

Based on this evidence, we schematically represent in Fig. 3 the phase-diagram of the model. Our simulations, performed up to size  $N=50\,000$ , seem to indicate that the parameter space is divided into genuine phases separated by well-defined transition lines. However, a detailed investigation of the nature of all of them (and of the associated critical behavior) is numerically very demanding and goes beyond the scope of the present paper.

Some of the features of the phase-diagram are recovered (see Fig. 3) via a mean-field (MF) approach, similar to the one in Ref. [13]. At each step, a node with  $k$  links is randomly chosen. Since the rewiring dynamics leads to a network with a Poisson degree distribution, the probability  $P_k(t)$  to extract a node with  $k$  links is supposed to be a Poisson distribution with mean  $\langle k \rangle$  at each step. Denote with  $j$  the number of unsatisfied links and with  $\mathcal{P}_{k,j}$  the probability that the chosen node is not stable and hence must be updated. With probability  $\phi \mathcal{P}_{k,j}$  a random unsatisfied link is rewired. In this case, the density of unsatisfied links changes by  $\Delta \rho = -\frac{2}{\langle k \rangle N}$ . On the contrary, with probability  $(1-\phi) \mathcal{P}_{k,j}$  the state

of the node is flipped and the density of unsatisfied links changes by the quantity  $\Delta\rho = \frac{2(k-2j)}{\langle k \rangle N}$ . Using these expressions, it is possible to write the time evolution master equation for a generic update rule

$$\begin{aligned} \frac{d\rho}{dt} &= \sum_k \frac{P_k}{1/N} \sum_{j=0}^k B_{k,j}^{(\rho)} \mathcal{P}_{k,j} \left[ (1-\phi) \frac{2(k-2j)}{\langle k \rangle N} - \phi \frac{2}{\langle k \rangle N} \right] \\ &= F_{\phi,s}(\rho), \end{aligned} \quad (1)$$

where  $1/N$  is the temporal interval between successive steps and  $B_{k,j}^{(\rho)}$  is the probability to find a node with  $j$  unsatisfied links. In a MF spirit, all nodes of the network can be considered equivalent and the probability to have an unsatisfied link can depend only on the global observable  $\rho$ . Thus, the probability to have an unsatisfied link is taken independent for each node and is well approximated by a binomial distribution  $B_{k,j}^{(\rho)} = \binom{k}{j} \rho^j (1-\rho)^{k-j}$ .

In the case of the voter model the update probability is simply  $j/k$ , and it is possible to analytically solve it [13]. In our model, the update probability is  $\mathcal{P}_{k,j} = \theta(j/k + s - 1)$ , where  $\theta(x)$  is the Heaviside step function. Due to the nonlinearity of Eq. (1) an analytical expression can be found only for  $s=1$ . In such a case the right-hand side of Eq. (1) has the simple expression

$$\begin{aligned} F_{\phi,s}(\rho) &= 2(1-\phi)(1-2\rho) - \frac{2}{\langle k \rangle} \phi \\ &\quad - e^{-\langle k \rangle \rho} \left( 2(1-\phi)(1-\rho) - \frac{2}{\langle k \rangle} \phi \right). \end{aligned} \quad (2)$$

For  $\rho_s$  sufficiently small, the stationary solution of Eq. (1) has then the form  $\rho_s = \frac{2[\langle k \rangle(1-\phi)-1]}{\langle k \rangle[2+\langle k \rangle(1-\phi)-3\phi]}$ . For  $\phi < \phi_c(s=1) = \frac{\langle k \rangle - 1}{\langle k \rangle}$  there is an active stationary state with  $\rho_s > 0$ . For larger values of  $\phi$  the density of unsatisfied links is zero, corresponding to an absorbing phase. The transition is predicted to be continuous. For  $\langle k \rangle = 10$  the critical value is  $\phi_c(s=1) = 0.9$ , in agreement with numerical simulations. It is possible to determine numerically the transition line  $\phi_c(s)$  for any value of  $s$ . The resulting curve is reported in Fig. 3.

The simple model we have proposed offers a surprisingly rich variety of possible scenarios by varying the two parameters  $s$  and  $\phi$ . In particular, phase boundaries correspond to magnetization, connectedness and/or percolation transitions. The most striking feature, absent in all other models of co-evolution, is the existence of a phase where stable homogeneous domains coexist in the system even if the latter is not split into components. This feature is due to the presence of the threshold  $s$ : models characterized by a threshold are likely to display this type of behavior and represent a promising option for a realistic description of social phenomena. We stress however that the goal of this paper was not a description of a specific real world phenomenon, but rather the investigation of what are the possible qualitative outcomes when threshold dynamics and rewiring operate simultaneously.

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